Progress in Lattice QCD for Exotics

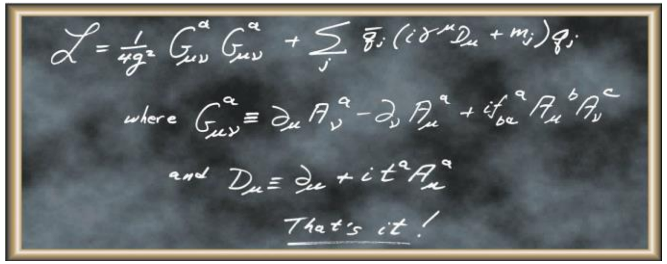
Nilmani Mathur

Department of Theoretical Physics

Tata Institute of Fundamental Research, India

Flavor Physics and CP violation (FPCP2022), May 23-27, 2022

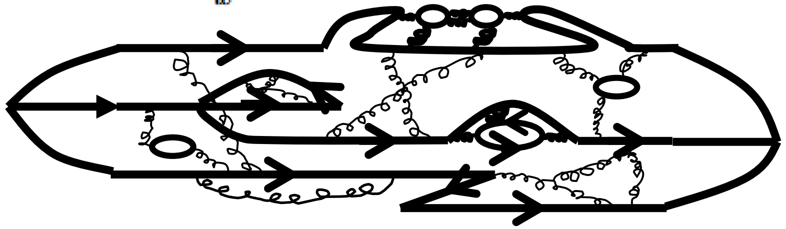
QCD



$$S_{QCD} = \int d^4x \, L_{QCD}(m_{q,}g_s)$$

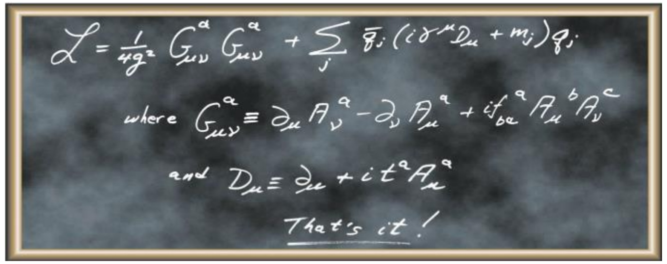
$$\langle C \rangle = \frac{\int DGDqD\bar{q}Ce^{-S_{QCD}}}{\int DGDqD\bar{q}\, e^{-S_{QCD}}}$$

$$C_{\mathcal{O}}(t_i, t_f) = \sum_{\vec{x}} e^{-i\vec{p}.\vec{x}} \langle 0|\mathcal{O}(\vec{x}_f, t_f)\bar{\mathcal{O}}(\vec{x}_i, t_i)|0\rangle$$

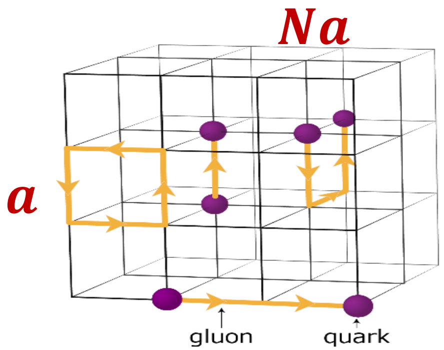


large time $\sim e^{-E_0 t}$

QCD → LQCD



Euclidean time



$$S_{QCD} = \int d^4x \, L_{QCD}(m_q, g_s)$$

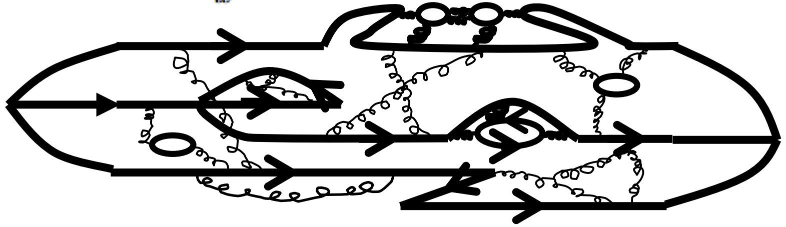
$$\langle C \rangle = \frac{\int DGDqD\bar{q} C e^{-S_{QCD}}}{\int DGDqD\bar{q} \, e^{-S_{QCD}}}$$

$$S^E_{QCD} = S^E_{QCD}[U, qi, D(U), m_{q_i}, a]$$

$$\langle C \rangle = \frac{\int DUDqD\bar{q} C e^{-S^E_{QCD}}}{\int DUDqD\bar{q} \, e^{-S^E_{QCD}}} \approx \frac{1}{N} \sum_n C(D^{-1}(U_n))$$

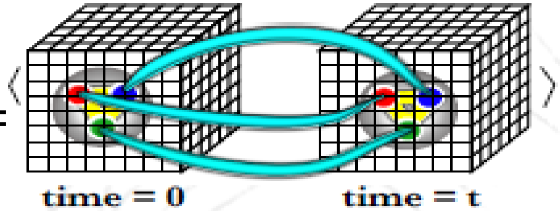
$$\Delta C = \frac{1}{\sqrt{N}} + \text{systematics}$$

$$C_{\mathcal{O}}(t_i, t_f) = \sum_{\vec{x}} e^{-i\vec{p}.\vec{x}} \langle 0 | \mathcal{O}(\vec{x}_f, t_f) \bar{\mathcal{O}}(\vec{x}_i, t_i) | 0 \rangle$$

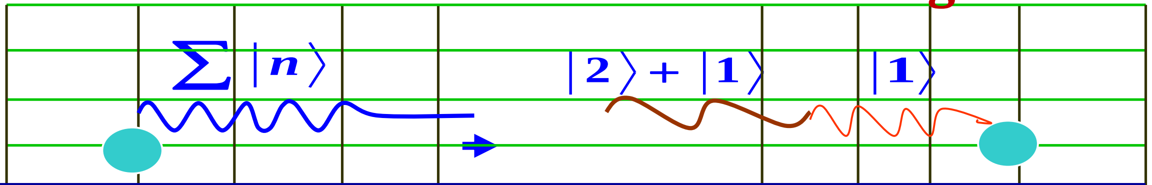


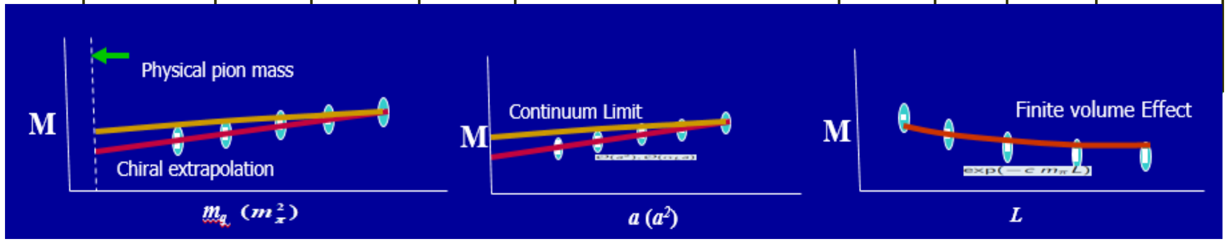
large time $\sim e^{-E_0 t}$

$$\left\langle C^{2pt}_{ab}(t, \vec{P}) \right\rangle = \langle \ldots \rangle = \sum_n Z_{b,n} Z^\dagger_{a,n} e^{-E_n t}$$

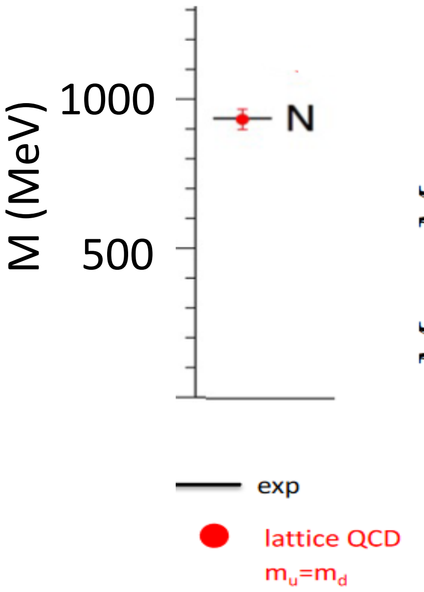


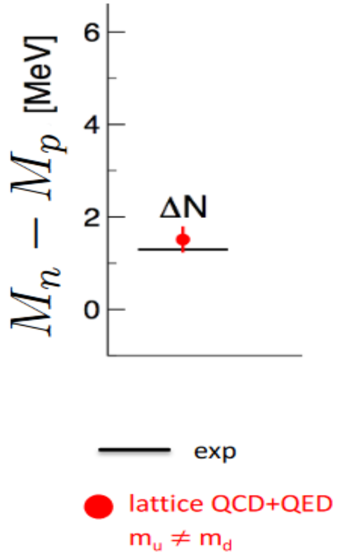
τ large $\sim e^{-E_0 \tau}$





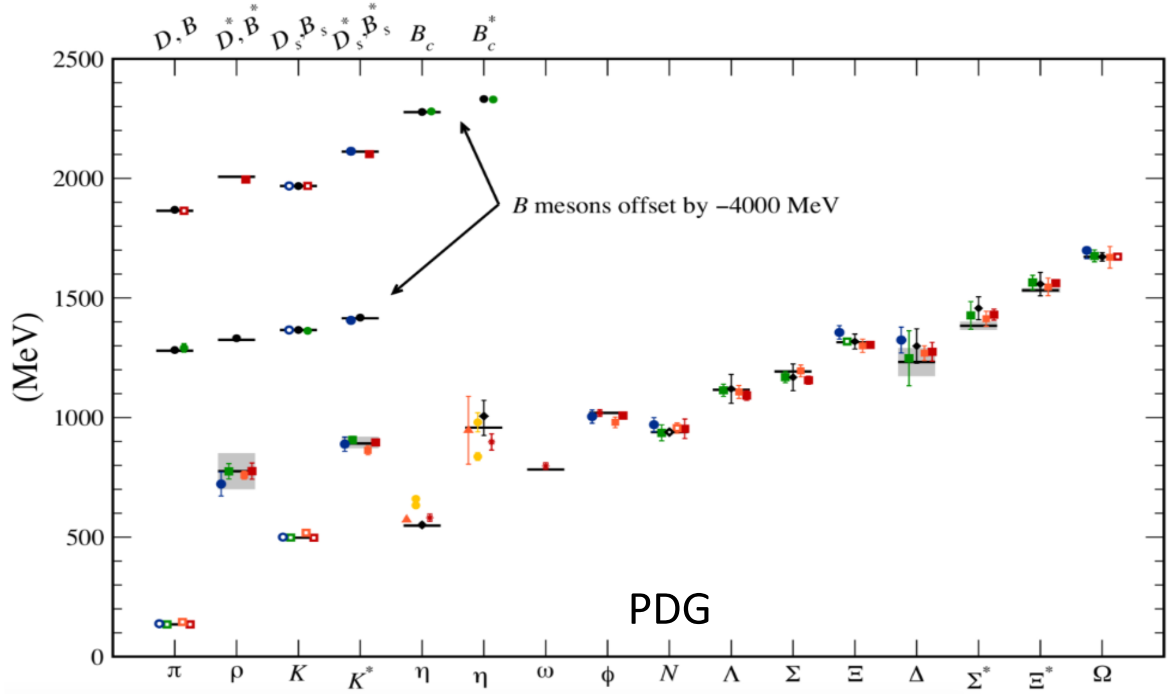
A few notable Lattice QCD results on hadron-spectroscopy





BMW Collaboration
Science 322, 2008

BMW Collaboration
Science 347, 2015



Hadron Spectra: What can Lattice QCD do?

Relatively Easy	Moderately difficult	Difficult
Threshold		
 Stable under strong interactions, Far away below threshold		

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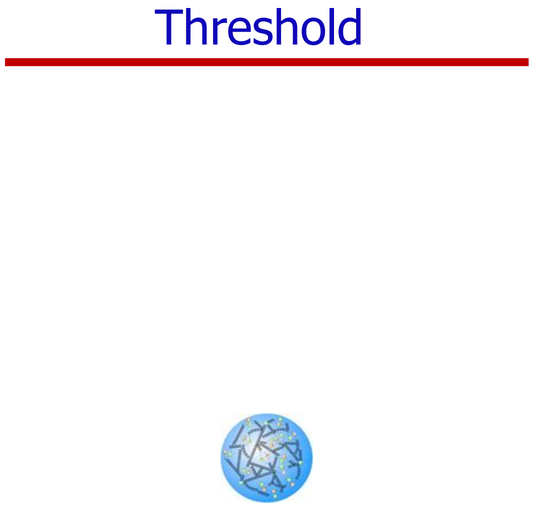
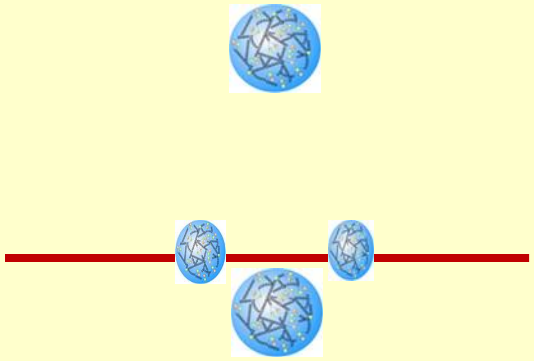
Relatively Easy



Stable under strong interactions,
Far away below threshold

Moderately difficult



- Close to threshold loosely bound,
- Resonances with two-body decay

Difficult

Hadron Spectra: What can Lattice QCD do?

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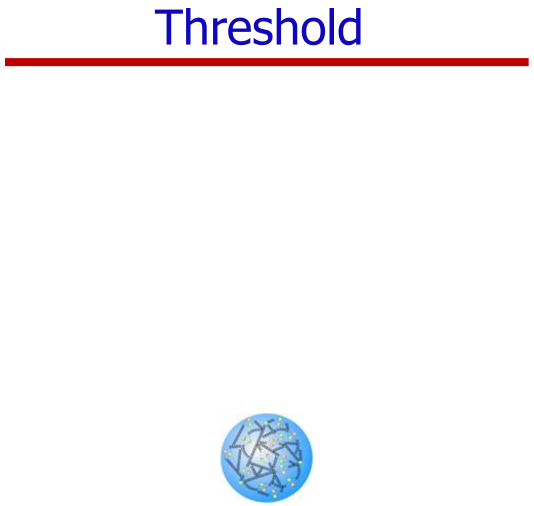

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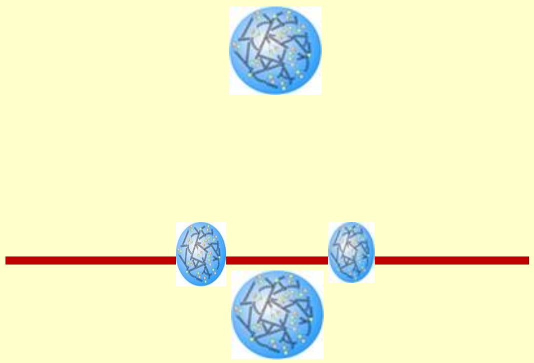

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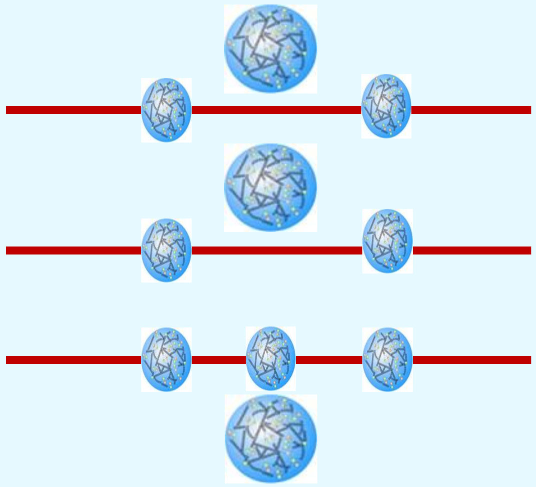

- Close to threshold loosely bound but has three or more body decays,
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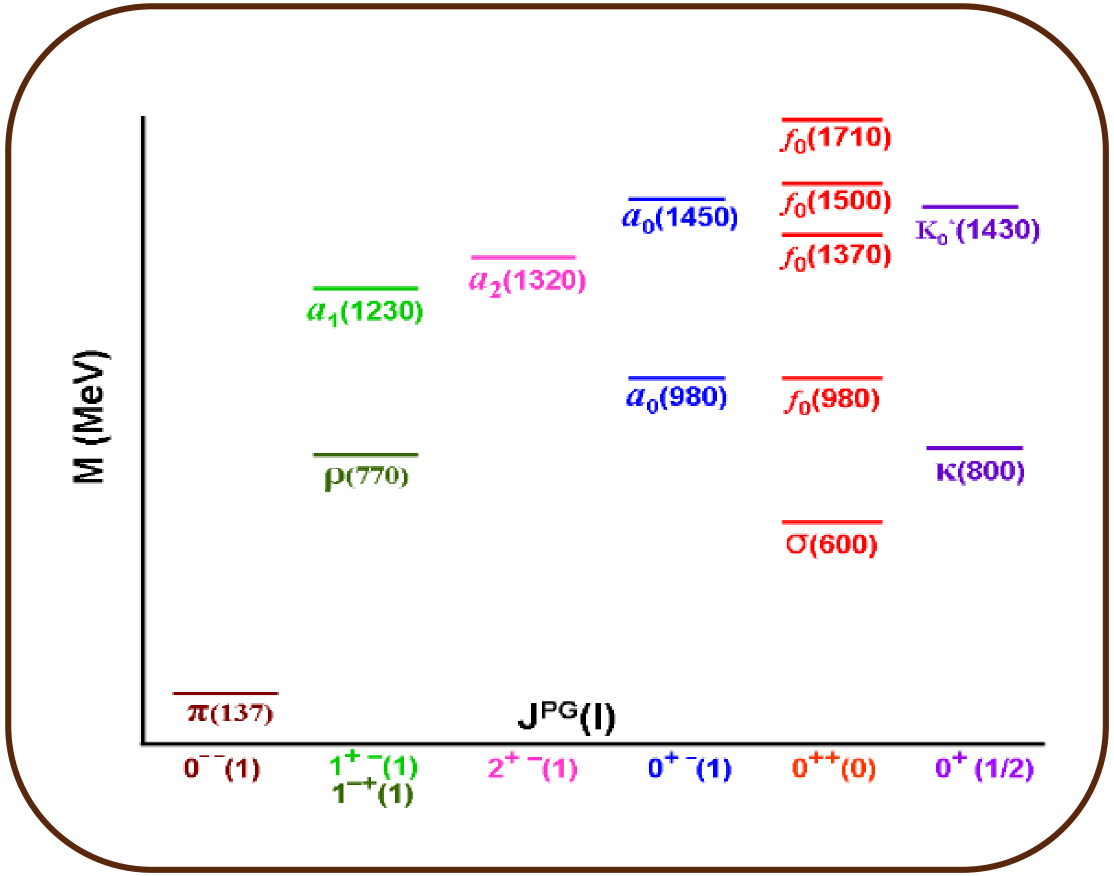
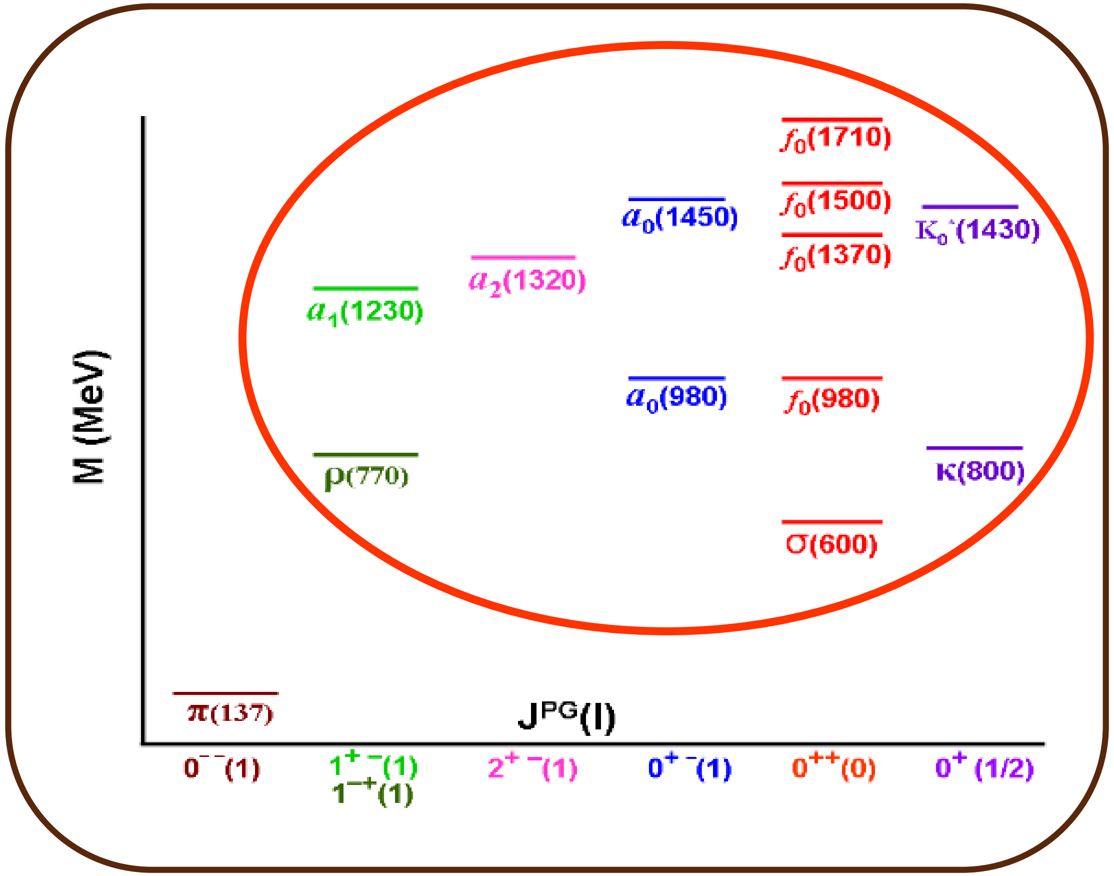
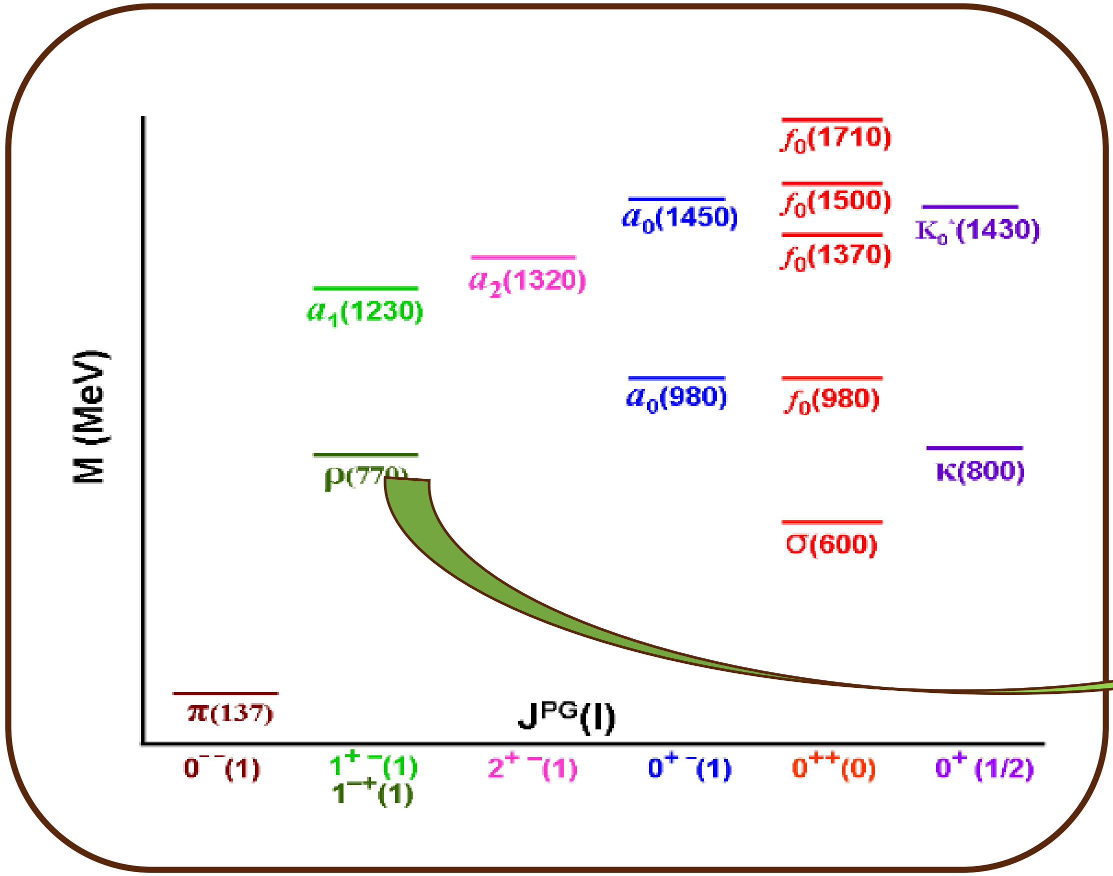


Low lying mesons



Almost all of them are resonances

Low lying mesons



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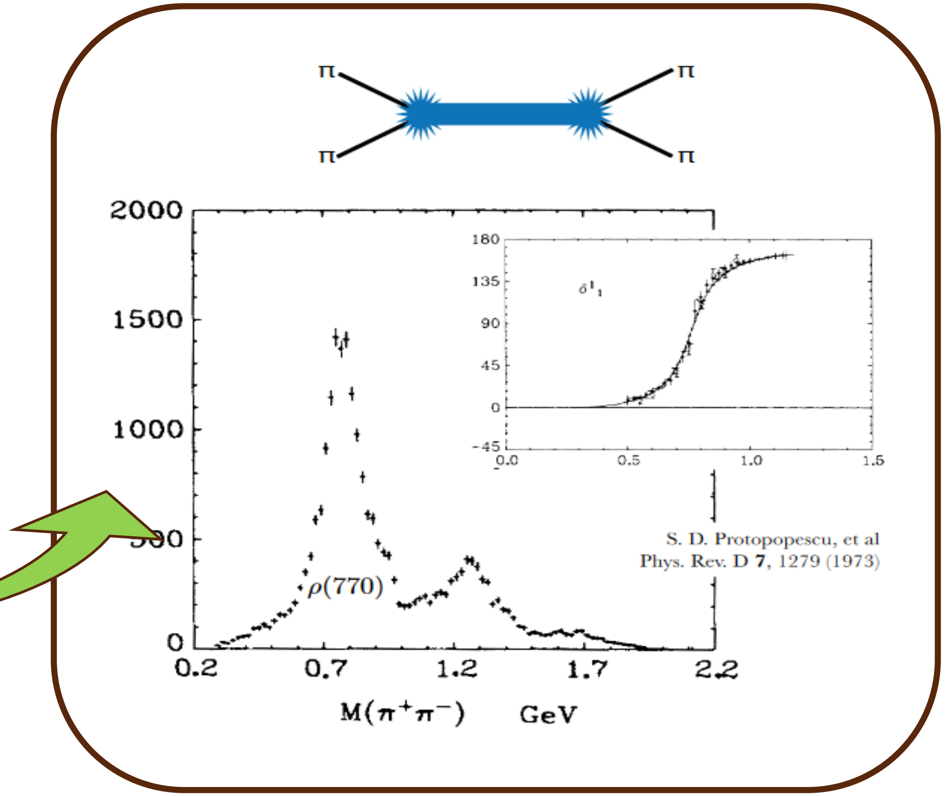
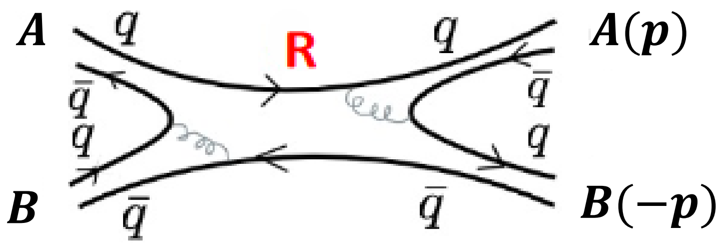


Almost all of them are resonances

Scattering study on continuum



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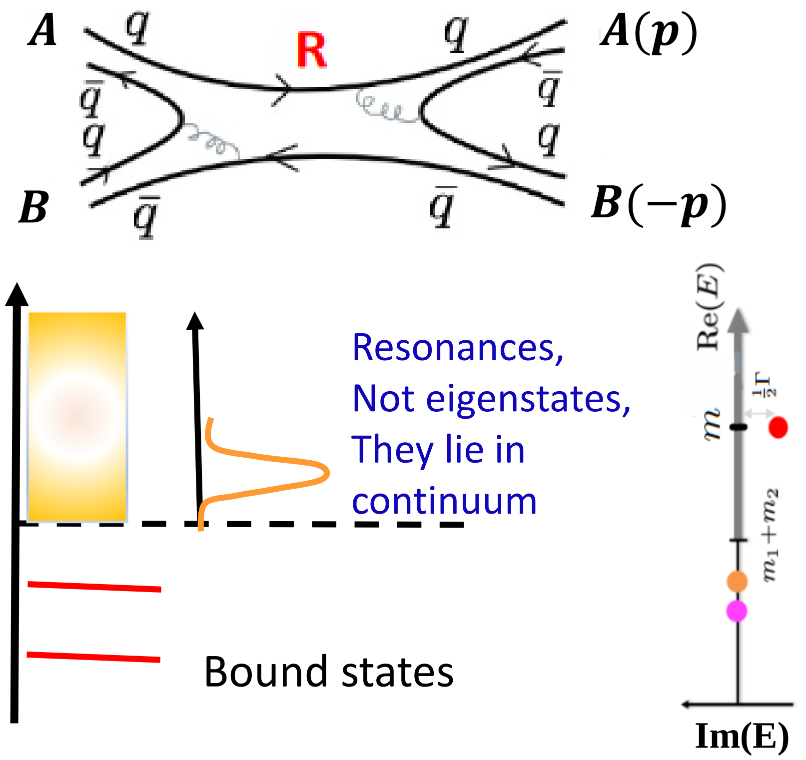
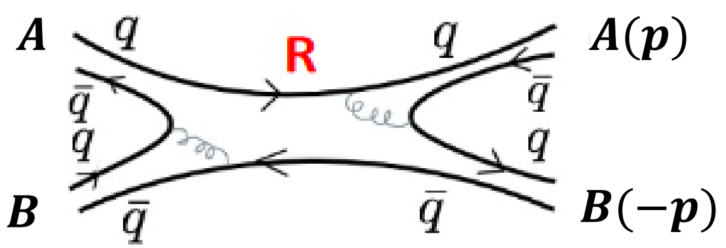
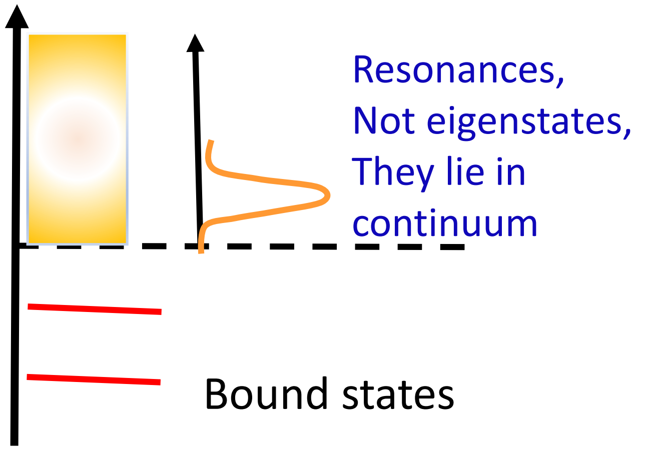
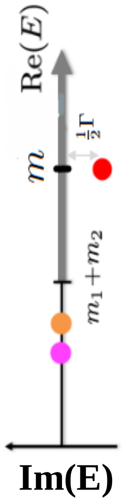


QM analogy: $\psi(|x| > R) \sim \cos(p|x|) + \delta(p))$

Scattering study on Lattice







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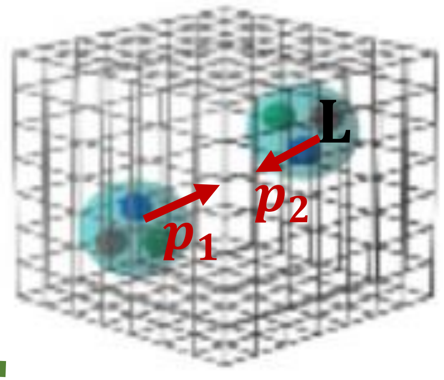

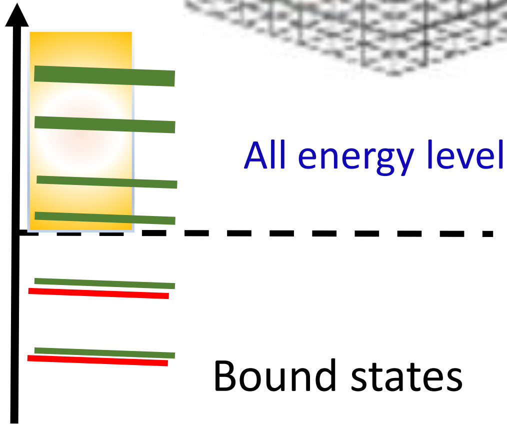

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using periodic boundary condition: $p_n(L) = \frac{2\pi}{L} n - \frac{2}{L}\delta(p)$

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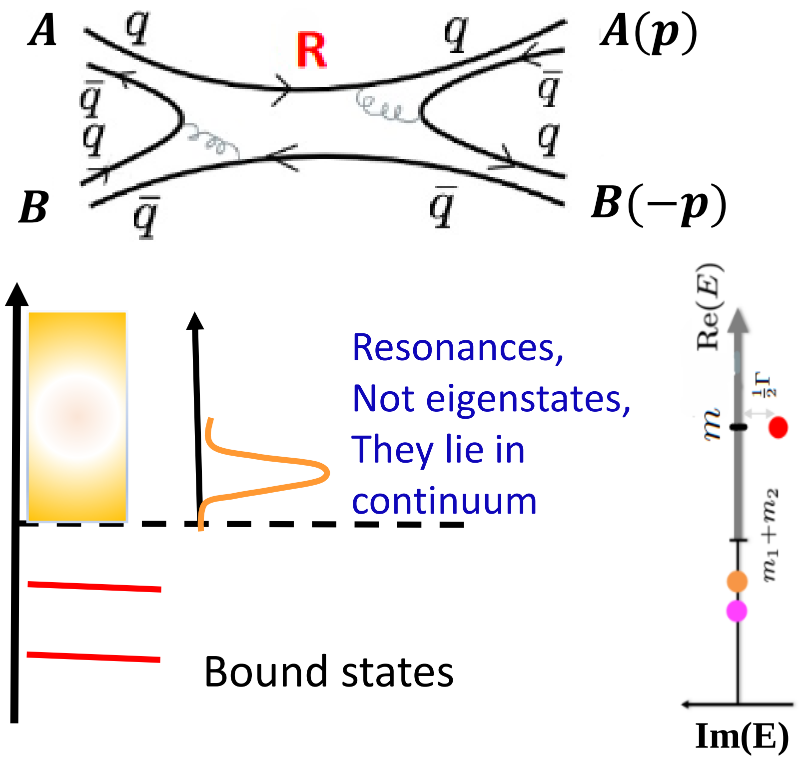

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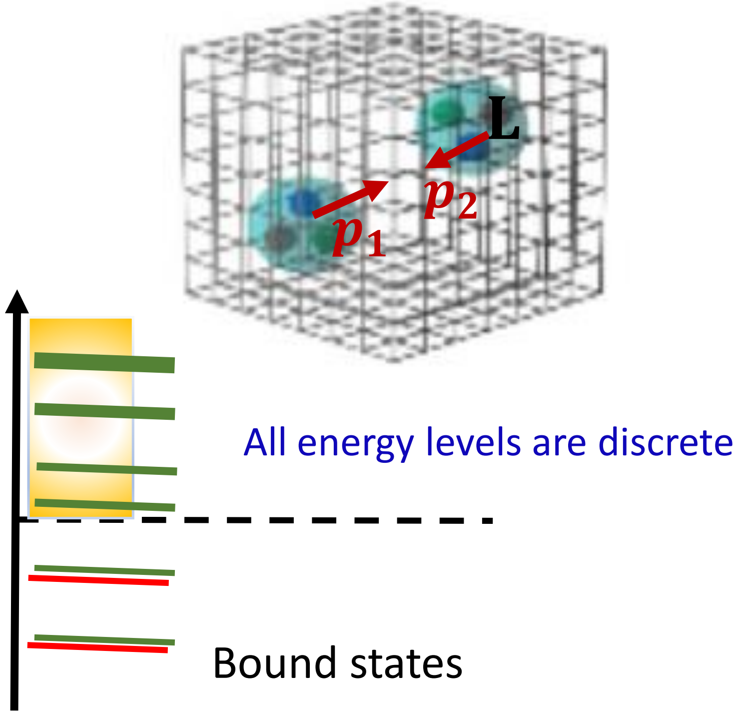

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Luscher's quantization condition for 2 → 2 scattering processes:

$$\det\left[M(E) + F^{-1}(E, p; L)\right] = 0$$

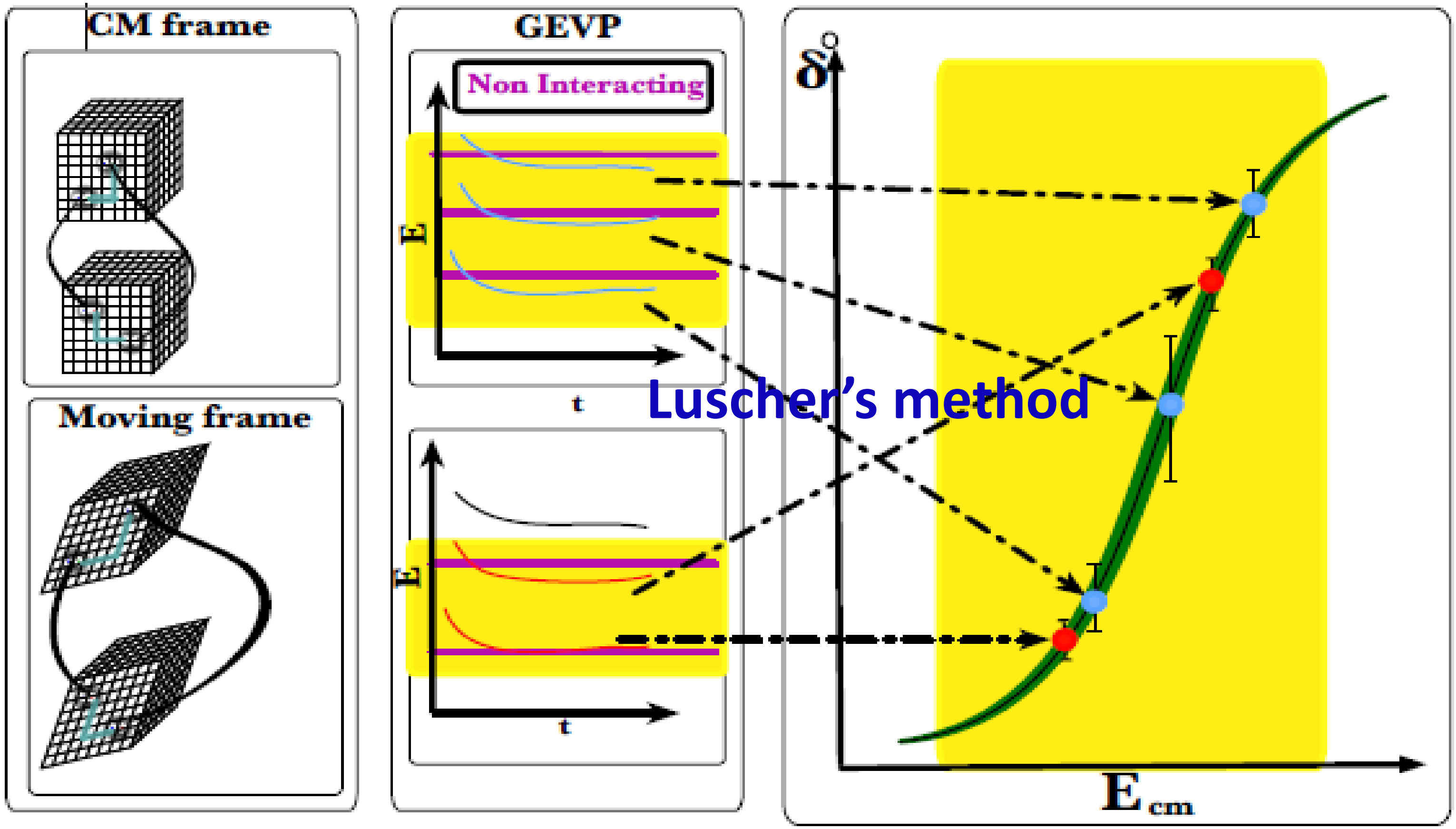
Infinite volume scattering amplitudes

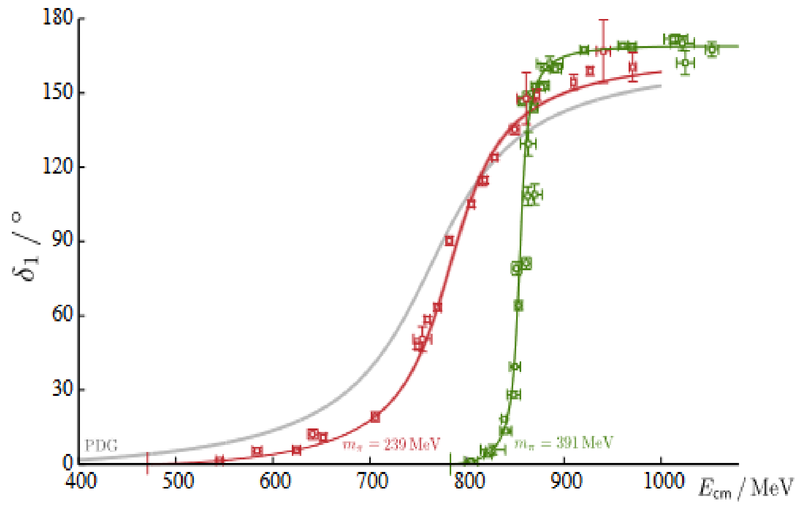
Finite volume spectra + boundary condition

Luscher (1991)

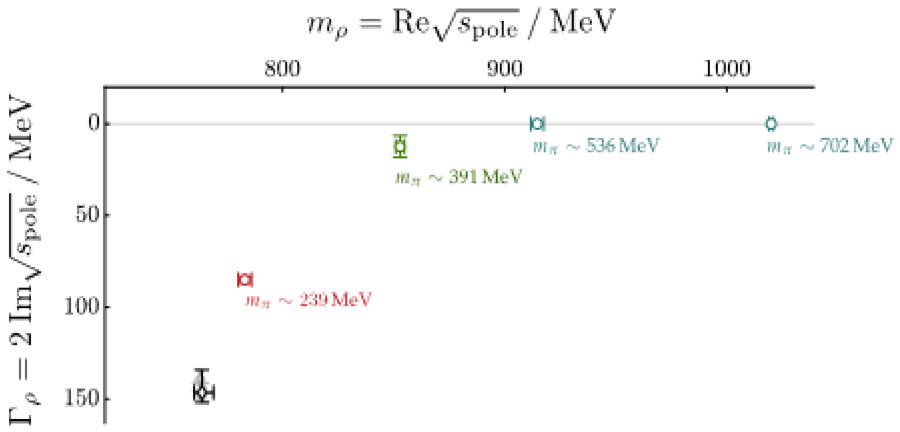
Recent review: *Rev.Mod.Phys.* 90 (2018) 2, 025001
Brecino, Dudek, Young

In a nutshell





Hadspec: PRD (2013, 2015)



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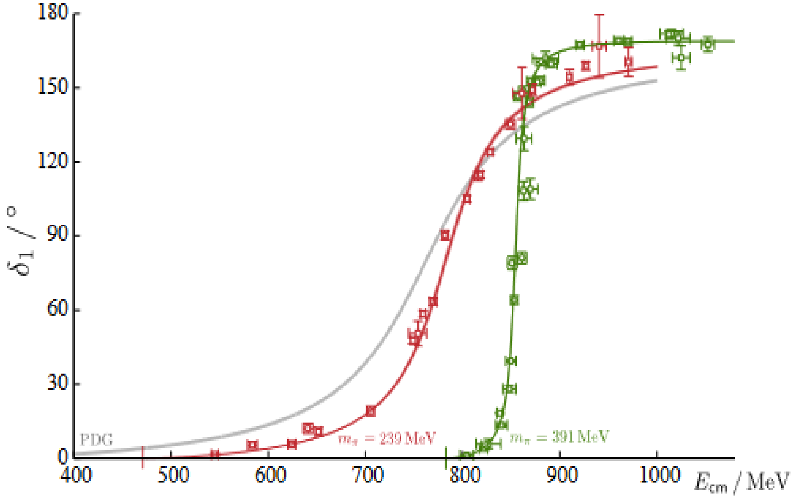

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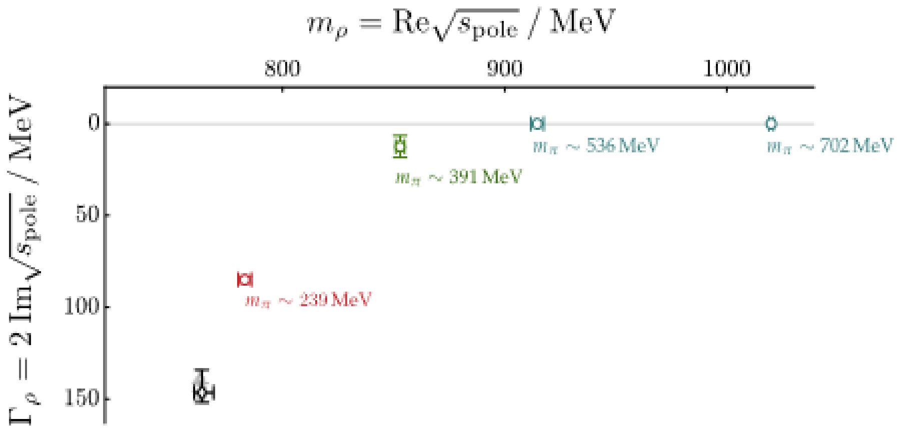

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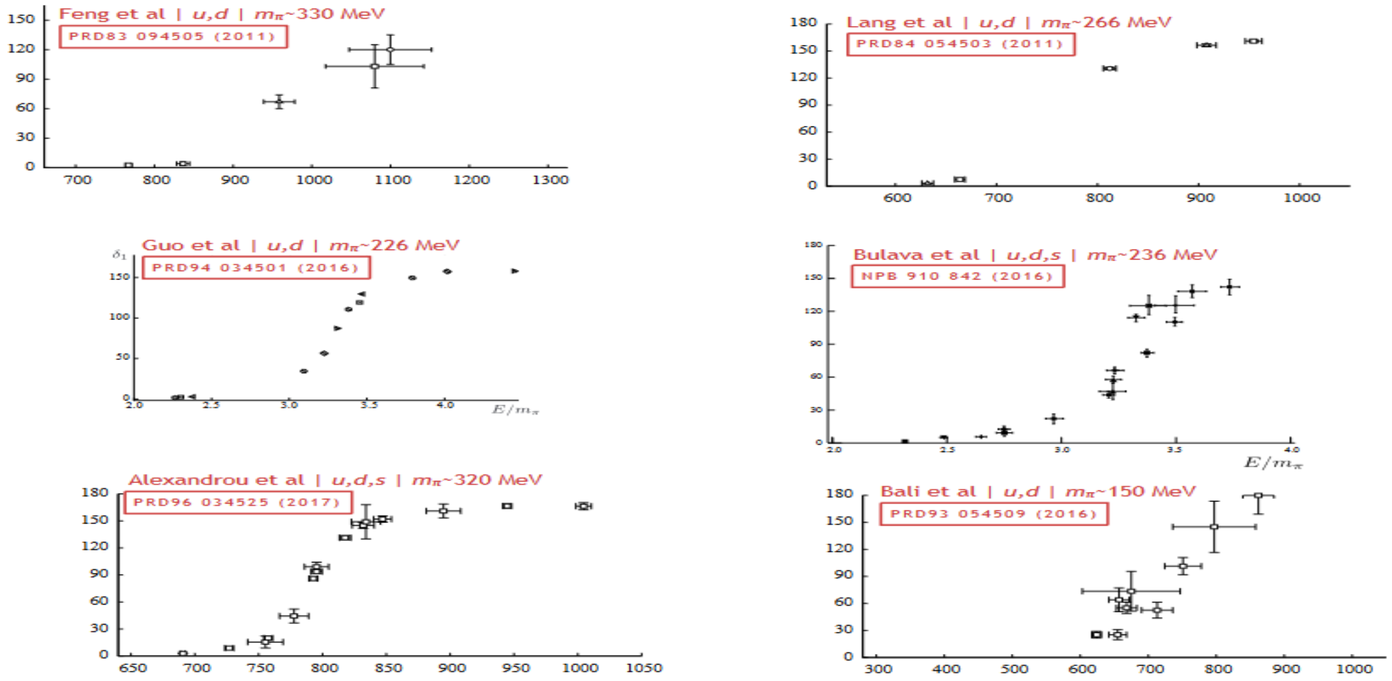

Where it currently stands

- Lattice QCD has demonstrated its efficacy in elastic scattering processes, especially for $\rho \to \pi\pi$

- Many coupled-channel cases have also been studied

  $\rho, \sigma, \kappa, \kappa^*, f_0, a_0, b_1, f_2, a_2$, a few **D mesons** have been studied with impressive results

  Hadspec collaboration;
  Morningstar, Bulava etal;
  Prelovsek, Padmanath, Mohler, Collin et al
  …and many others (apology for not mentioning all names)

- Three body decays are under study

- For precision study: Needs multiple lattices including large volumes and finer lattices
  Computer intensive → Next 5 years?

Heavy Four-quark states

- Fourquark states have been observed experimentally with heavy quark contents. ….LHC, Belle, BES
- Are there possibilities to find more of those? And other multiquark states?
- What can lattice studies do?

LQCD for heavy quark physics

Requirement: lattice quark mass $ma << 1$ **discretization error**

$$\frac{1}{L} \ll m_\pi \ll m_H \ll \frac{1}{a}$$

Computational cost $\propto \frac{1}{a^{4-6}}$ **!**

Lattice QCD study of heavy-light hadron requires lattices with smaller lattice spacings as well as larger volumes

Much more computer intensive for rigorous study

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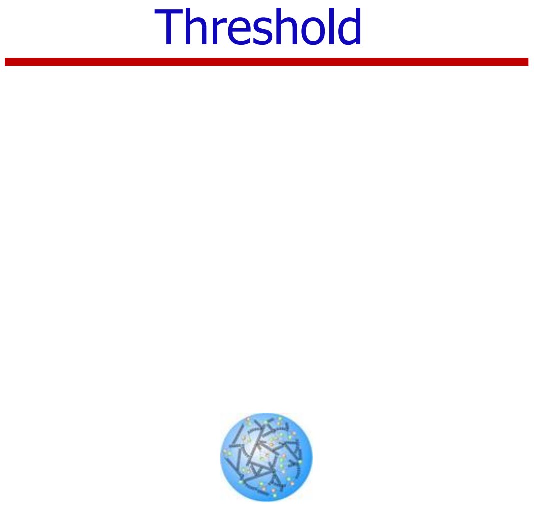
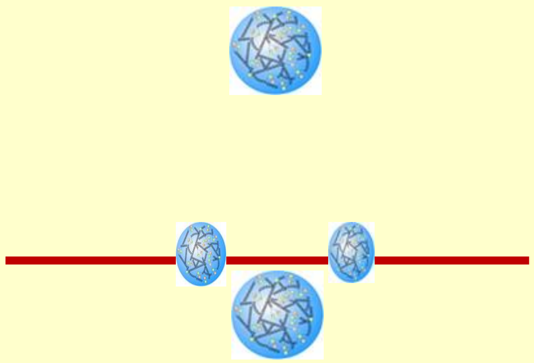


Stable under strong interactions,
Far away below threshold

Moderately difficult



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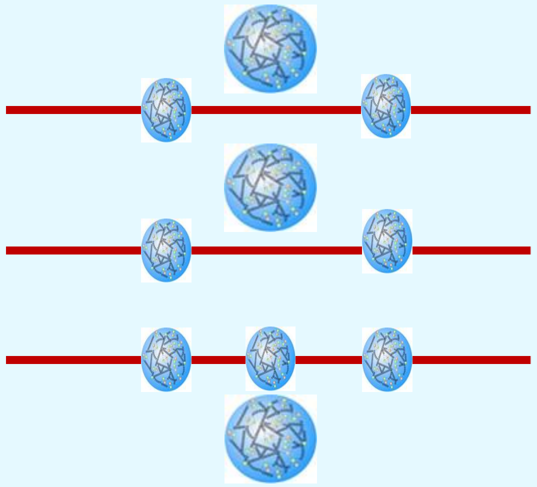

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Lattice study of heavy exotics

- Nonrelativistic *b* quark with relativistic other quarks (calculations with relativistic *b* quarks are starting)

- Potential:
  - Static quark potential (Born-Oppenheimer)…

    P. Bicudo et al, Phys. Rev. D95, 034502(2017)
  - HALQCD lattice potential ..HAL QCD ('16,'18)

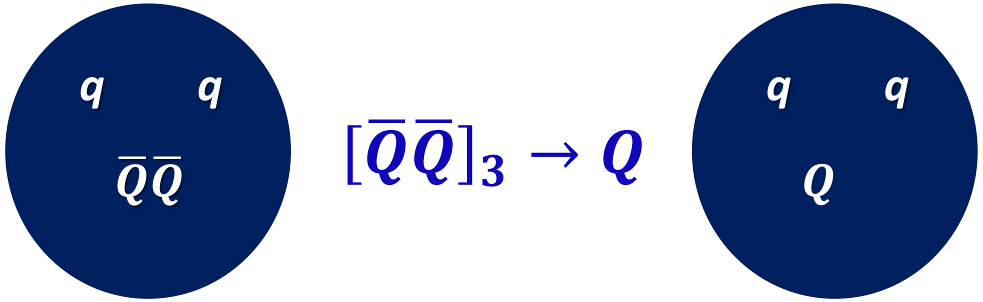
Interpolating operators

Tetraquark type: $[\psi_i(x)\psi_j(x)][\bar{\psi}_l(x)\bar{\psi}_k(x)]$

Meson-meson type: $\left(\sum_x e^{ip_1 \cdot x}\ \bar{\psi}_k(x)\Gamma\psi_i(x)\right)\left(\sum_y e^{ip_2 \cdot y}\ \bar{\psi}_l(y)\Gamma\psi_j(y)\right)$

with the appropriate color and spin combinations

Heavy four-quark states



Diquark properties:

- $[\bar{Q}\bar{Q}]_3 \xrightarrow{m_Q \to \infty}$ compact .
- $\{qq'\}_{\bar{3}}$ **attractive** .
- $\{qq'\} \equiv (qC\gamma_5 q')$ **lightest** .
- $m(\{ud\}) < m(\{us\})$.

A possible structure:

$(qC\gamma_5 q')(\bar{Q}C\gamma_i\bar{Q}')$

How about ? $\{qq'\}$ $\{\bar{Q}\bar{Q}'\}$

How to build heavy tetraquarks? A way

- Two heavy quarks with two light quarks
- $C_l = \bar{3}$, good light diquark
  - ➢ $F = \bar{3}, J_l = 0 \Rightarrow J_h = 1, C_h = 3, \; J^P = 1^+$

$$(qC\gamma_5 q')(\bar{Q}C\gamma_i\bar{Q}')$$

$$\{\bar{3}, J = 0\} \quad \{3, J = 1\}$$

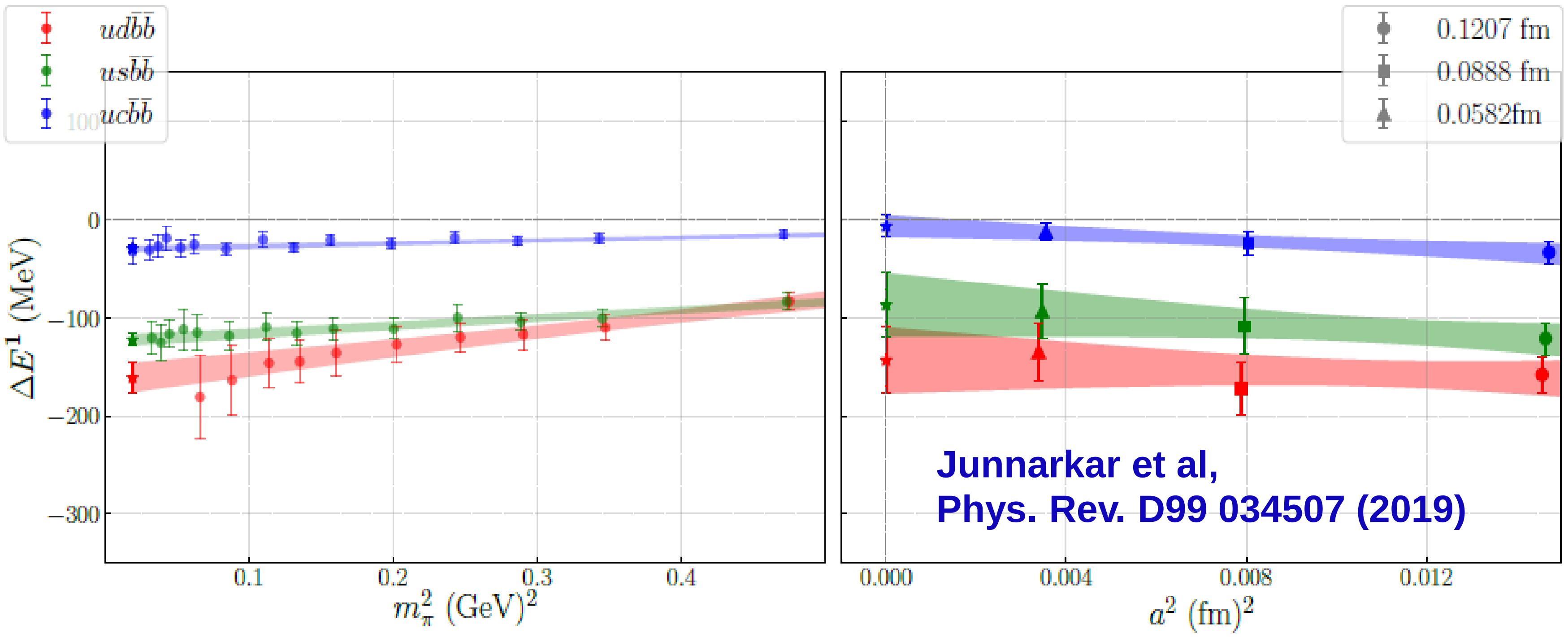
- ➢ Spin dependent interaction $\propto 1/m_h$. For threshold $J^P = 1^+$ states, like $B^*B, B^*D, B_s^*B_s, D^*D$, this interaction expected to be supressed.

- ➢ With $C_h = 3$, colour Coulomb attraction, this is not present for two-meson thresholds.

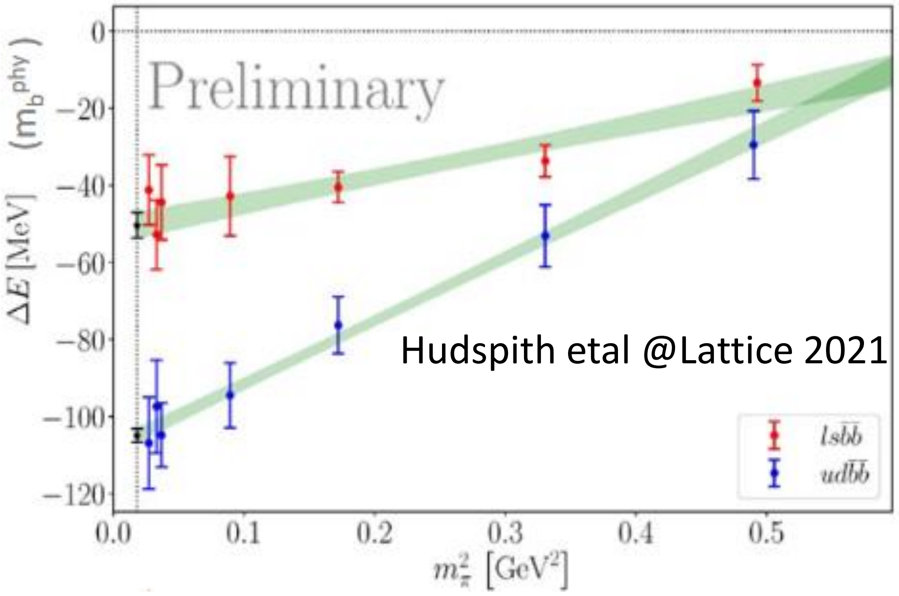
Possible states? : $\bar{b}\bar{b}ud, \bar{b}\bar{b}us, \bar{b}\bar{b}uc, \bar{b}\bar{b}sc,$
$\bar{b}\bar{c}ud, \bar{b}\bar{c}us \;\; etc.$

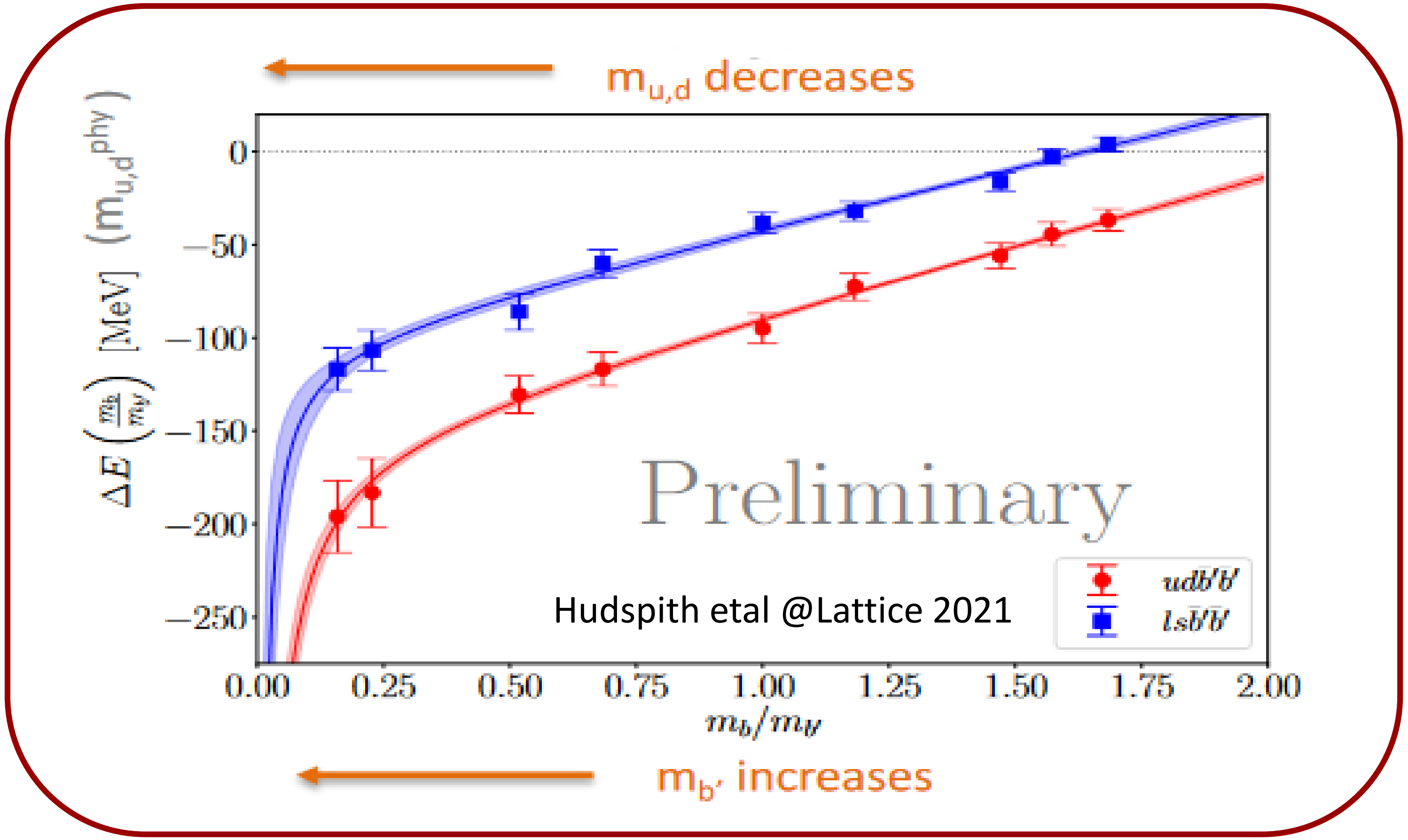
$$J = 1, l_1 l_2 \bar{Q}\bar{Q} \qquad\qquad J = 0, ll\bar{Q}\bar{Q}$$

@Francis et al (2016)

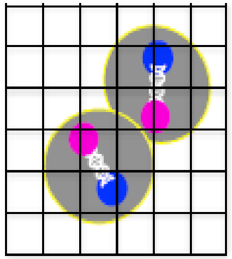


$\bar{b}\bar{b}q_1q_2$



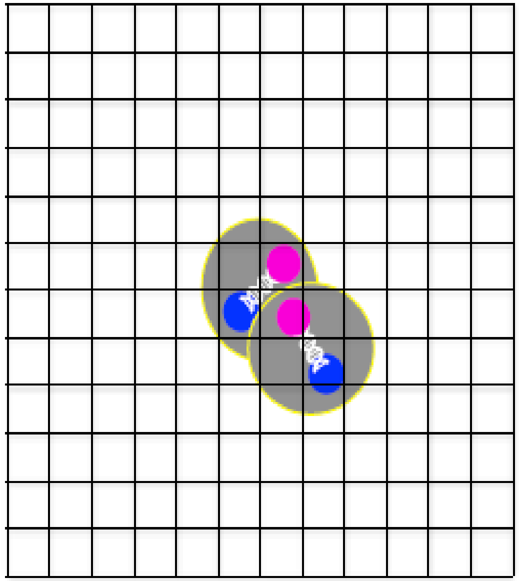


- **Heavier the heavy quark masses, deeper the binding**
- **Lighter the light quark masses, deeper the binding**



Infinite-volume
bound state

$$[E(L) - E(\infty)] \propto \frac{e^{-\gamma L}}{\gamma L}$$



Huang, Yang, Phys. Rev. 105 (1957)

Lüscher, Commun. Math. Phys. 105 (1986)

Beane et al, Phys. Lett. B 585 (2004)

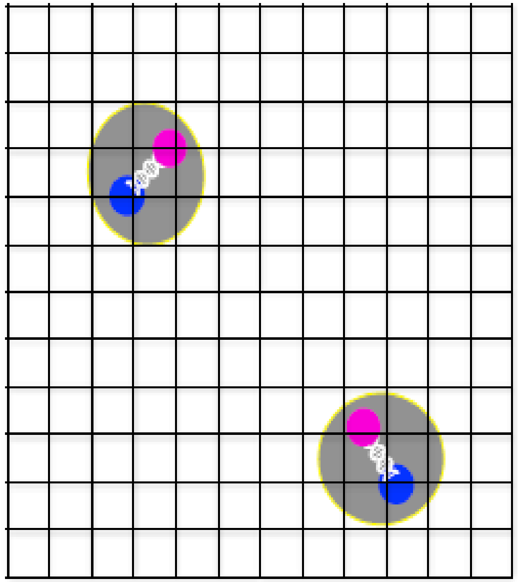
…

Recent review:

Briceño, Dudek, Young, Rev. Mod. Phys. 90 (2018)

Infinite-volume
scattering state

$$[E(L) - E(\infty)] \propto \frac{a}{ML^3}$$

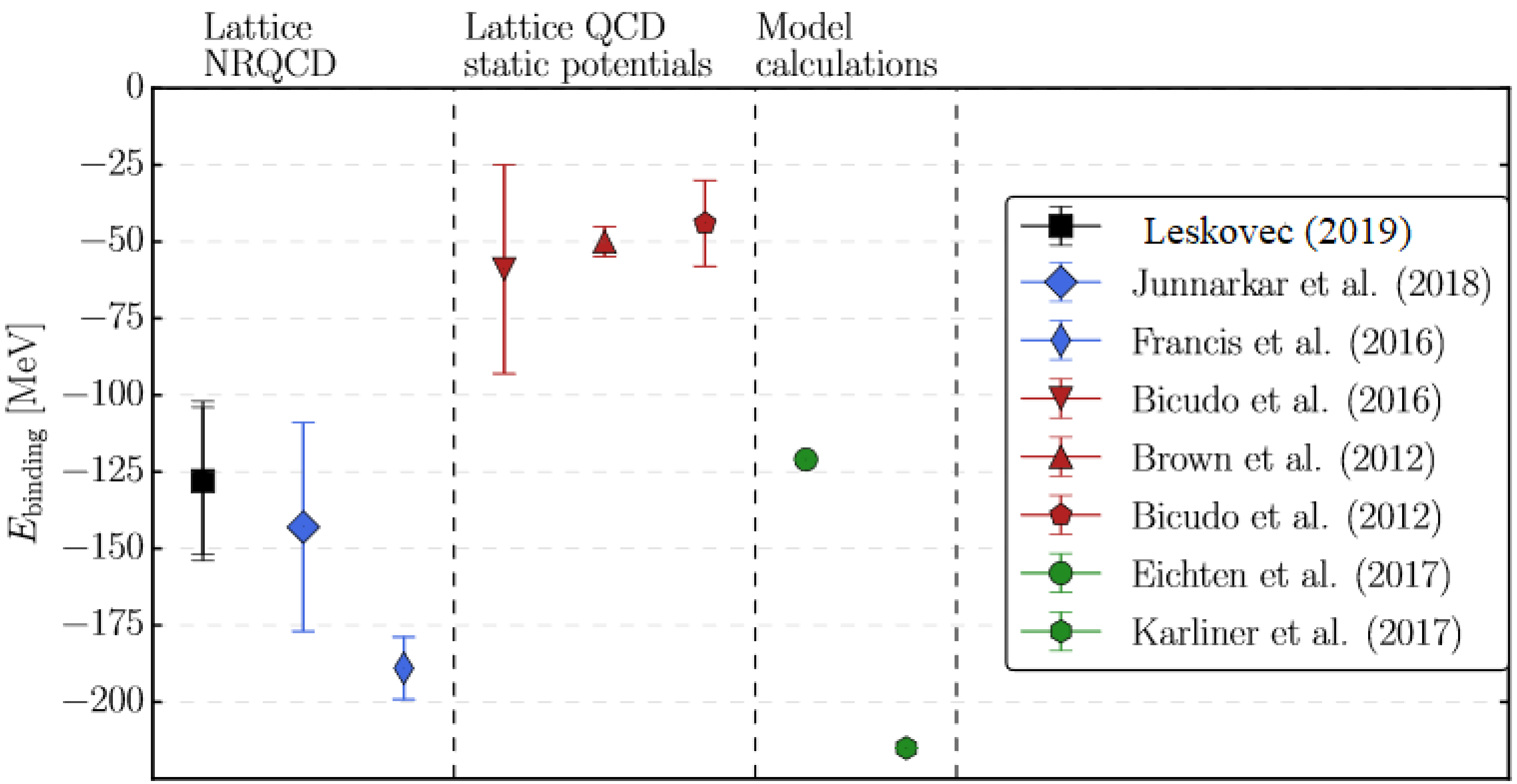


Finite volume effects for heavy hadrons:

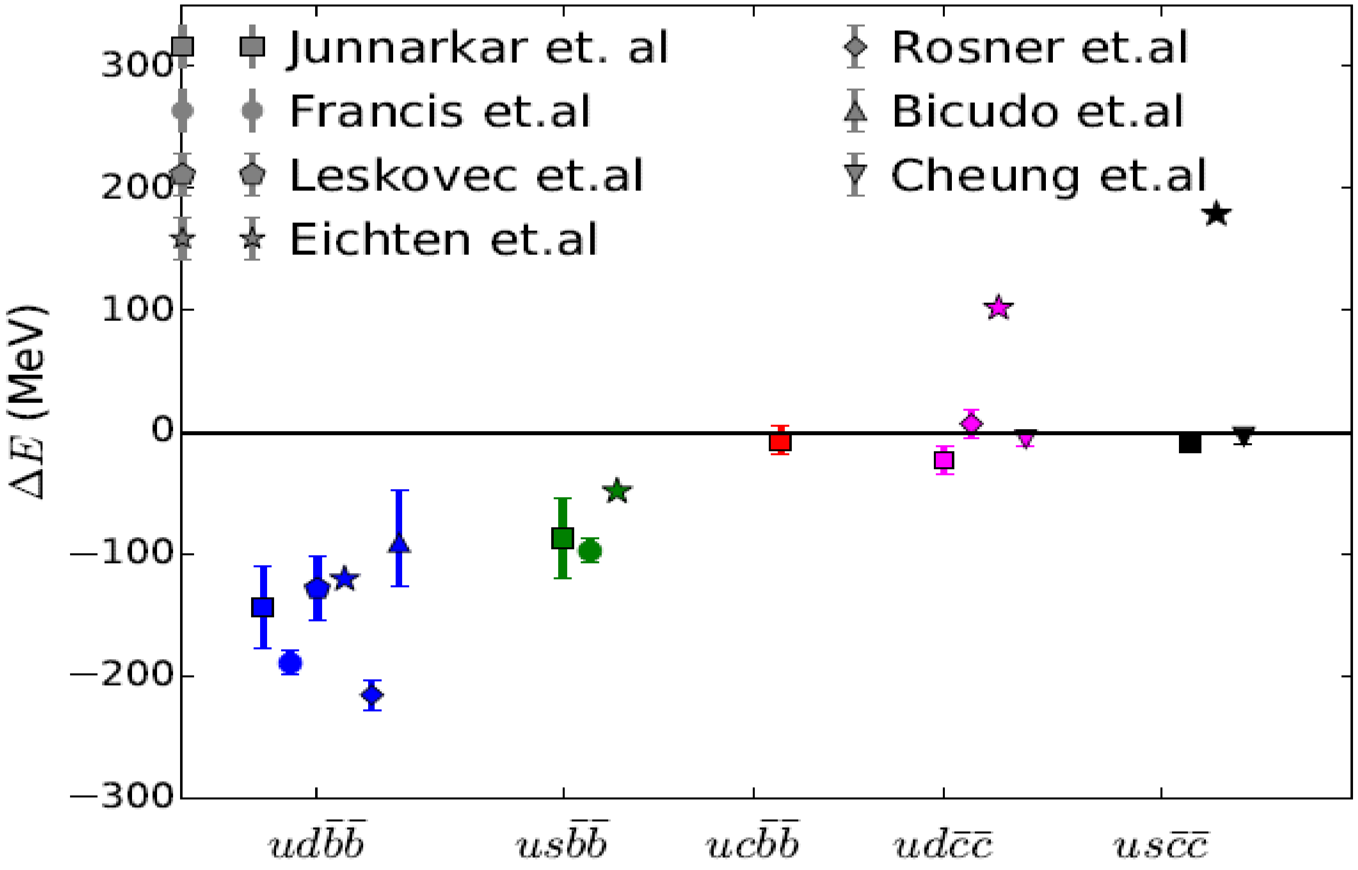
$$\Delta_{FV} = E_{FV} - E_\infty \propto \mathcal{O}(e^{-k_\infty L})/L,$$
$$\text{with} \quad k_\infty = \sqrt{(m_1 + m_2)B_\infty}\,,$$

Beane et al: Phys. Lett. B585 (2004)
Davoudi etal : Phys. Rev. D84 (2011) 114502

$ud\bar{b}\bar{b}$ (1^+)



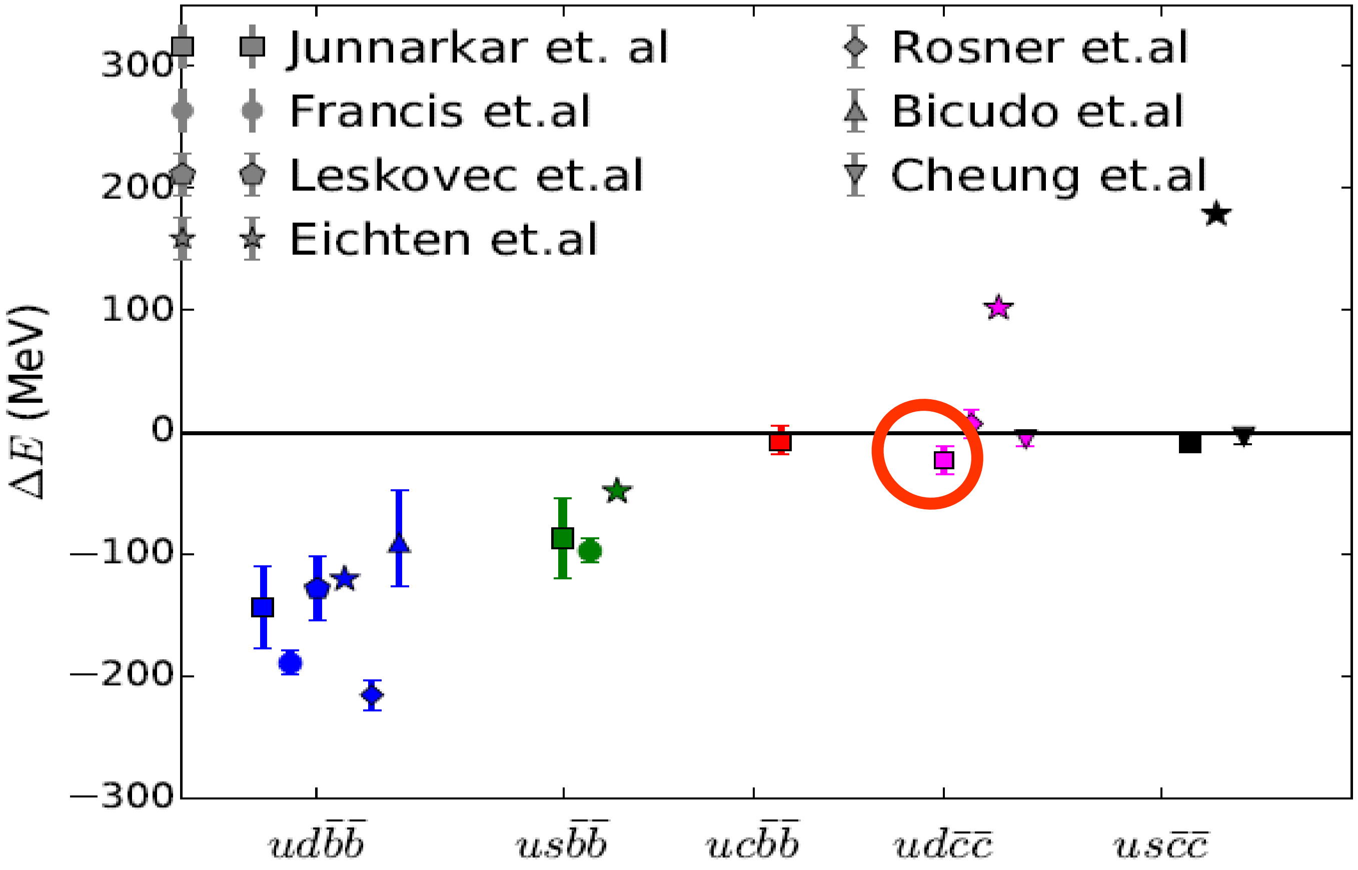
Leskovec et al, Phys. Rev. D 100, 014503 (2019)



Junnarkar et al,
Phys. Rev. D99 034507 (2019)

State	ΔE^1 [MeV]	State	ΔE^1 [MeV]
$ud\bar{b}\bar{b}$	-143(34)	$us\bar{b}\bar{b}$	-87(32)
$uc\bar{b}\bar{b}$	-6(11)	$sc\bar{b}\bar{b}$	-8(3)
$ud\bar{c}\bar{c}$	-23(11)	$us\bar{c}\bar{c}$	-8(8)

State	ΔE^0 [MeV]	State	ΔE^0 [MeV]
$uu\bar{b}\bar{b}$	-5(18)	$uu\bar{c}\bar{c}$	26(11)
$ss\bar{b}\bar{b}$	3(9)	$ss\bar{c}\bar{c}$	14(4)
$cc\bar{b}\bar{b}$	16(1)		



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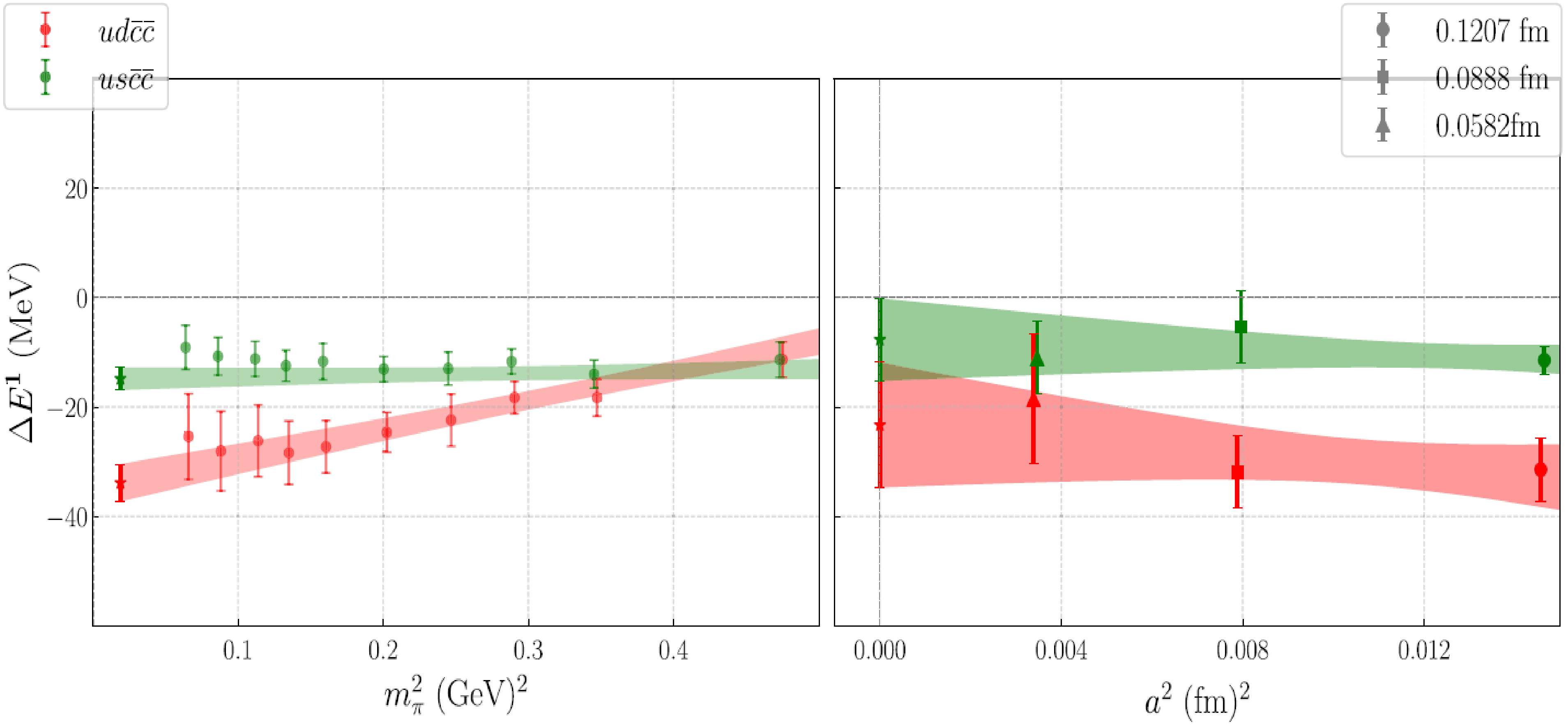
$ud\bar{b}\bar{b}$	-143(34)	$us\bar{b}\bar{b}$	-87(32)
$uc\bar{b}\bar{b}$	-6(11)	$sc\bar{b}\bar{b}$	-8(3)
$ud\bar{c}\bar{c}$	-23(11)	$us\bar{c}\bar{c}$	-8(8)

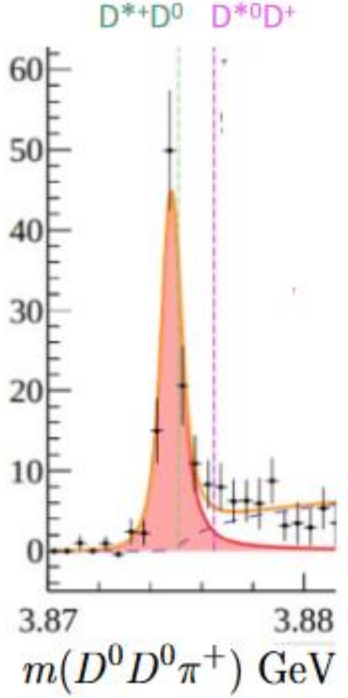
State	ΔE^0 [MeV]	State	ΔE^0 [MeV]
$uu\bar{b}\bar{b}$	-5(18)	$uu\bar{c}\bar{c}$	26(11)
$ss\bar{b}\bar{b}$	3(9)	$ss\bar{c}\bar{c}$	14(4)
$cc\bar{b}\bar{b}$	16(1)		

$\bar{c}\bar{c}q_1q_2$



T_{cc}

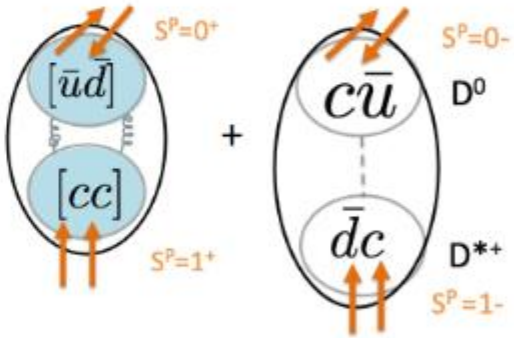
Doubly charm tetraquark T_{cc} $cc\bar{d}\bar{u}$



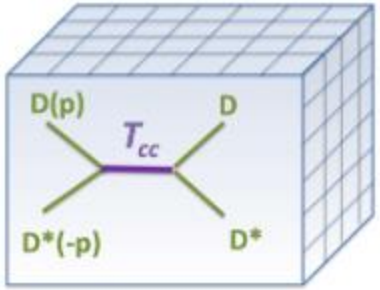
$$\delta m = m - (m_{D^{*+}} + m_{D^0})$$

$$\delta m_{pole} = -0.36 \pm 0.04 \text{ MeV}$$

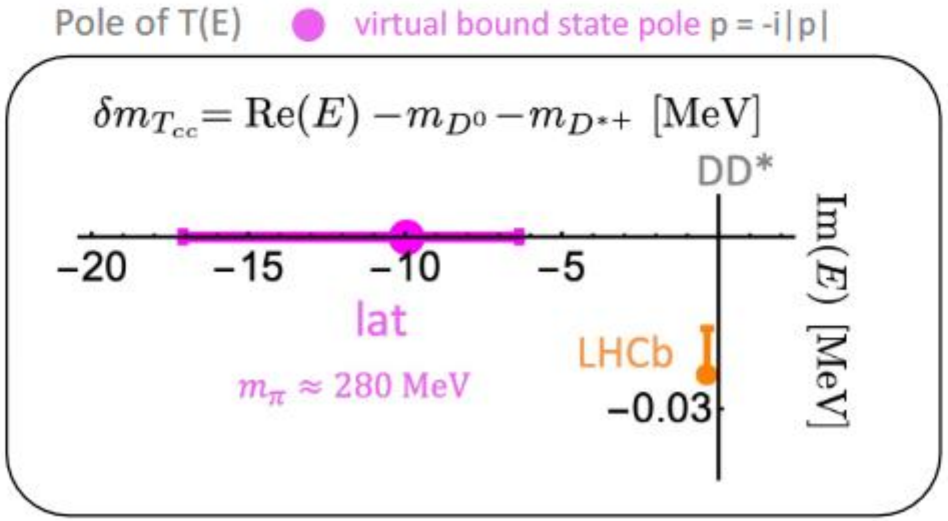
LHCb 2109.01038, 2109.01056

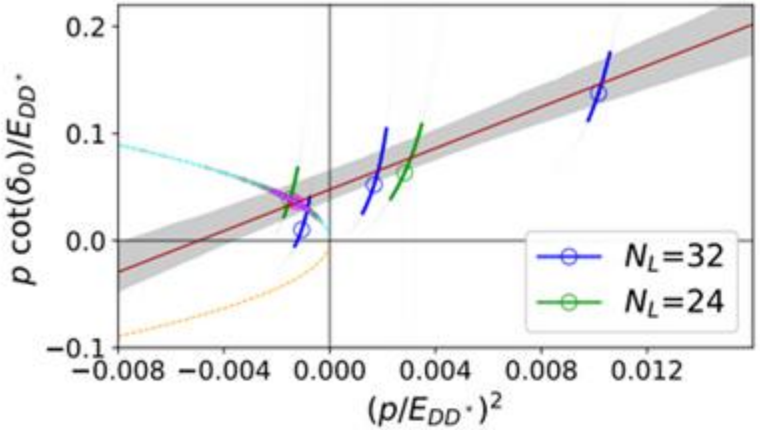


T_{cc} on lattice



arxiv: 2202.101101
Padmanath and Prelovsek

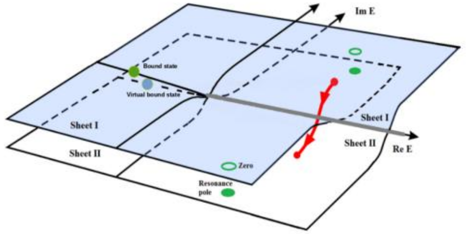




$$\delta m \equiv m_{T_{cc}^+} - (m_{D^{*+}} + m_{D^0})$$

	m_D [MeV]	$\delta m_{T_{cc}}$ [MeV]	T_{cc}
lat. $m_c^{(h)}$	1927(1)	$-9.9^{+3.6}_{-7.2}$	virtual bound st.
lat. $m_c^{(l)}$	1762(1)	$-15.0(^{+4.6}_{-9.3})$	virtual bound st.
exp.	1864.85(5)	−0.36(4)	bound st.

$\delta m = E^p_{cm} - E_{th}$



$$T \propto (p\cot\delta_0 - ip)^{-1}$$

- Bound state: $p = i|p| \rightarrow e^{ipr} = e^{-|p|r}$
- Virtual bound state $p = -i|p| \rightarrow e^{ipr} = e^{|p}$
  like the spin-singlet dineutron

Slide thanks: Padmanath and Sasa

What about $T_{bc}: \bar{b}\bar{c}q_1q_2$?

Various models predicted mixed results for $ud\bar{b}\bar{c}\ (1^+)$:

- HQ-symmetry inspired and non-chiral models: mostly unbound or very weekly bound
- QCD sum rule, chiral models: a bound state (both for 0 and 1-isospins) with binding over a wide range ~ 20-400 MeV !

Hudspith et al, Phys. Rev. D102, 114506 (2020)

Lattice QCD: $ud\bar{b}\bar{c}\ (1^+)$

$-61\ \mathrm{MeV} < \ \Delta E_{ud\bar{b}\bar{c}} \sim -15\ \mathrm{MeV}$

$\bar{D}B^*$threshold

A. Francis et al:

Phys. Rev. D99 (2019) no.5, 054505

No state below thresholds: Hudspith et al, Phys. Rev. D102, 114506 (2020), Lattice'21.
Pflaumer, Lattice'21

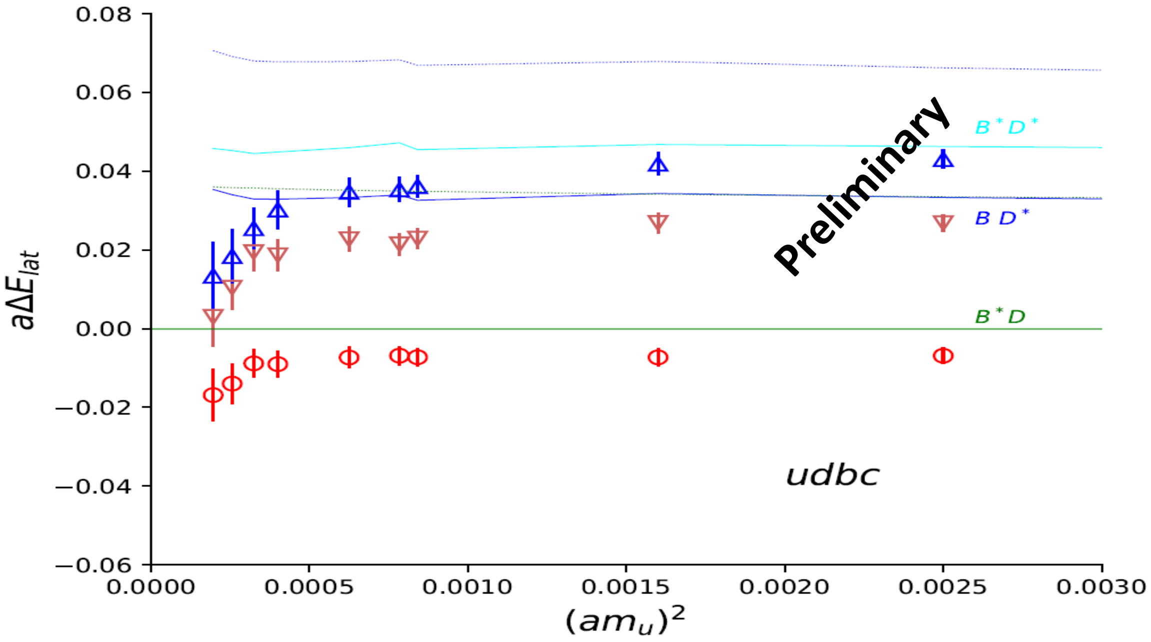
$ud\bar{b}\bar{c}\ (1^{+})$

New:

- We use more number of operators including also the diquark-antidiquark type operators with the light diquark having a spin 1 keeping in mind that the charm quark is much lighter compared to the bottom quark.

Preliminary Findings:

- Multiple energy levels around elastic thresholds.
- At least one energy level below the lowest two-meson threshold.



- Using only meson-meson operators on finer (~ 0.058 fm) lattice
- Need to do analysis with the full operator set
- Near closeness to thresholds demands a finite volume analysis to conclude

NM, Padmanath: arXiv:2111.01147

More work is in progress on to cross-check this finding with more statistics and another volume ---stay tuned

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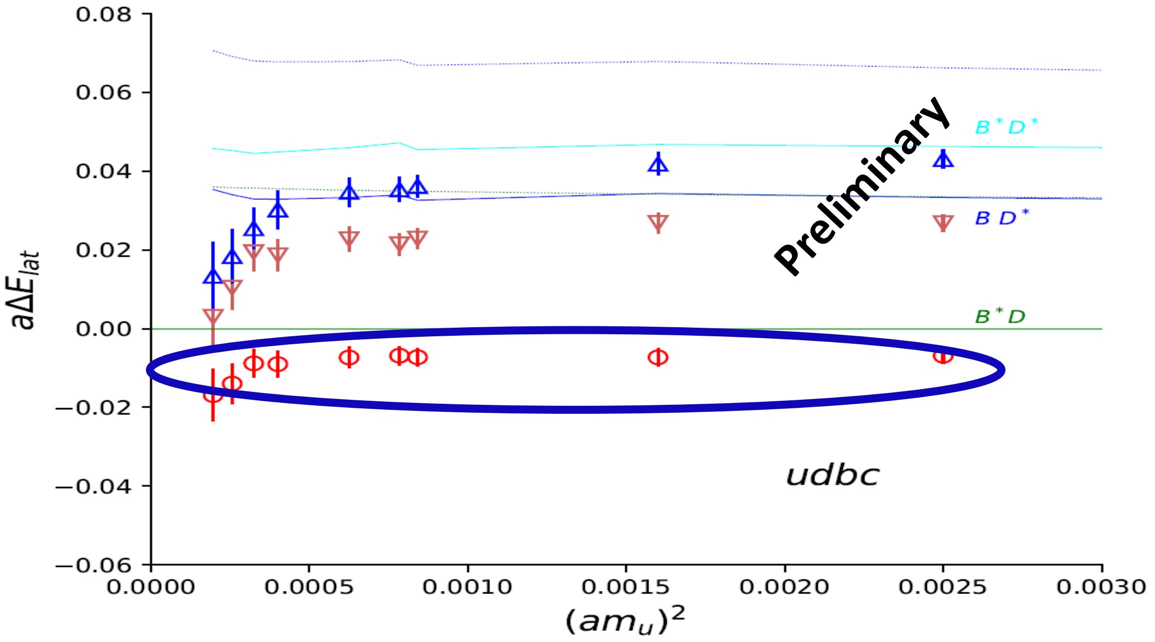

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X, Y, Z-type four quarks (LHCb, Belle, BES)

$$Q_i \bar{Q}_j q_l \bar{q}_k$$

Ex: $Z_b = b\bar{b}u\bar{d}$, $Z_c = c\bar{c}u\bar{d}$ etc.

$$Q_i \bar{Q}_j q_k \bar{q}_l \rightarrow Q_i \bar{q}_l + \bar{Q}_j q_k$$

$$Q_i \bar{Q}_j q_k \bar{q}_l \rightarrow Q_i \bar{Q}_j + \bar{q}_l q_k$$

Pentaquarks: LHCb (2015,2019)

$$Q_i \bar{Q}_j q_k q_l q_m$$

Ex: $P_c = c\bar{c}uud$

$$Q_i \bar{Q}_j q_k q_l q_m \rightarrow Q_i \bar{Q}_j + q_k q_l q_m$$

$$Q_i \bar{Q}_j q_k q_l q_m \rightarrow q_m \bar{Q}_j + Q_i q_k q_l$$

$$Q_i \bar{Q}_j q_k q_l q_m \rightarrow q_k \bar{Q}_j + Q_i q_l q_m$$

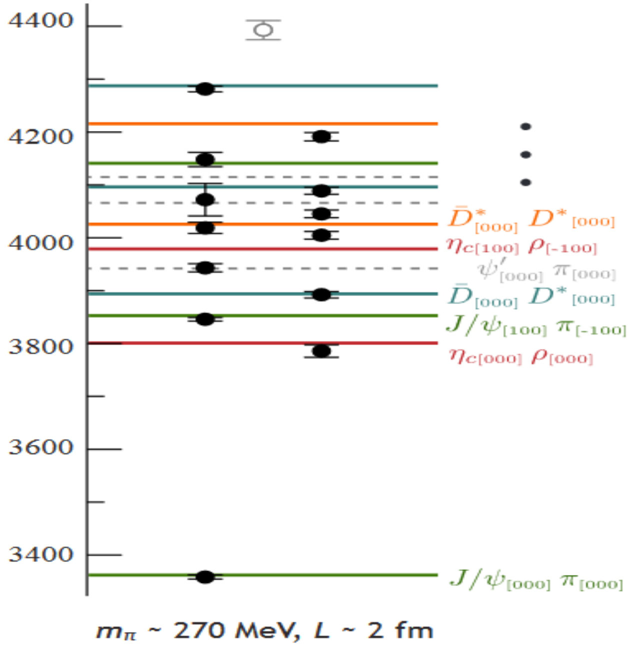
$$Q_i \bar{Q}_j q_k q_l q_m \rightarrow q_l \bar{Q}_j + Q_i q_k q_m$$

- Multiple decay modes
- Close to thresholds
- Presence of disconnected diagrams
- Both heavy and light quark together (multiple lattices with bigger volumes and smaller lattice spacings)

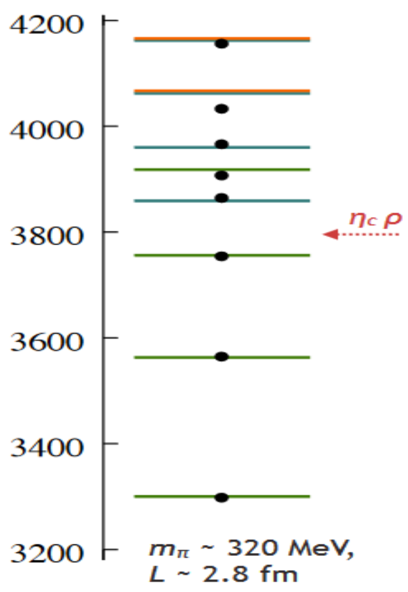
Difficult for lattice QCD precision study

$Z_c\ (J^P = 1^+)$ on lattice

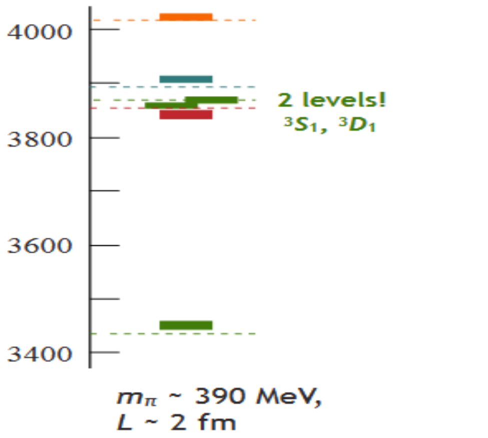
S. Prelovsek etal.:
Phys.Rev.D 91 (2015) 1, 014504



Chen et al:
Chin.Phys.C 43 (2019) 10, 103103



Cheung et al (Hadspec):
JHEP 11 (2017) 033



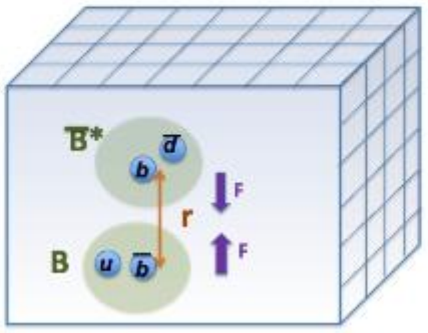
It is a resonance in a couple channel problem where one needs to deal with:

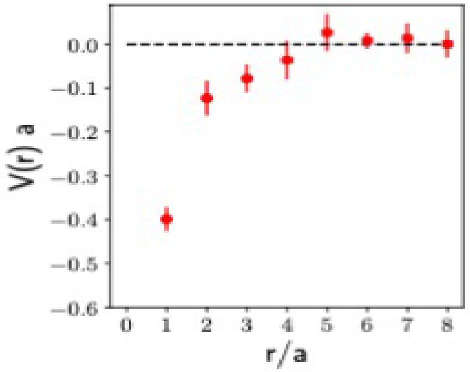
$$J/\psi\pi\ (^3S_1, {}^3D_1), DD^*(^3S_1, {}^3D_1), \eta_c\rho\ \rightarrow\ \eta_c\pi\pi\ (^3S_1, {}^3D_1)$$

- It's a difficult problem for lattice QCD considering so many channels
- Lattices with multiple volumes (large) and lattice spacings (as small as possible) with both charm and light quarks, and a large statistics are required for a precision study.

$Z_b = b\bar{b}u\bar{d}$ Belle(2011)

$b\bar{b}u\bar{d} \rightarrow BB^*, \Upsilon\pi$



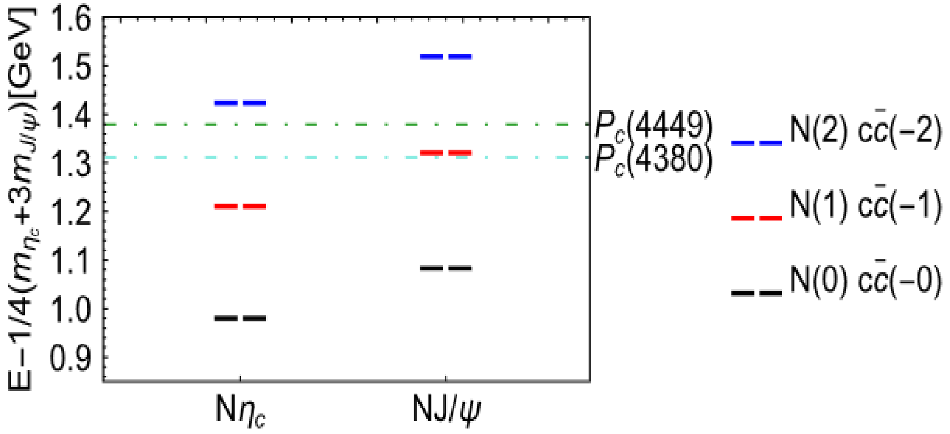


- Born-Openheimer approach with static b-quarks
- Attraction between B and $B^*$was found

Prelovsek et al: Phys. Lett. B 805 (2020) 135467

Bicudo et al: Phys. Rev. D 101, 034503 (2020)

$P_c = c\bar{c}uud$ LHCb (2015, 2019)



No interaction was found in NJ/ψ channel

Skerbis et al: Phys. Rev. D 99, 094505 (2019)

Light hybrid meson from 1^{-+} Lattice QCD

$\bar{d}\,G\,u \quad J^{PC} = 1^{-+}$

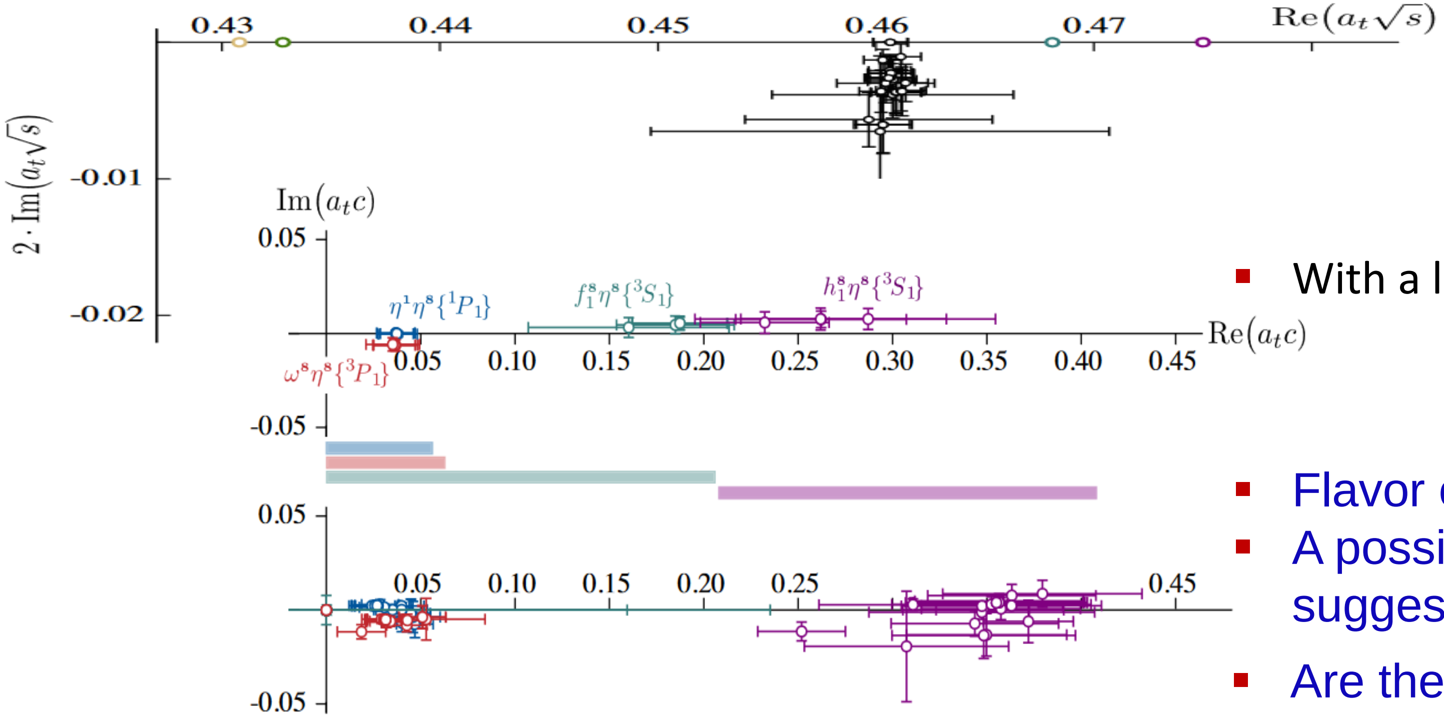
- Multiple decay modes: $\rho\pi, \eta\pi, \eta'\pi, f_1\pi, b_1\pi$ …
- First ever calculation of an exotic hybrid meson as a resonance in QCD

Hadspec collaboration:
PHYSICAL REVIEW D 103, 054502 (2021)

- Studies done at the $SU(3)_F$ point $m_u = m_d = m_s$; $(m_\pi = 700$ MeV$)$

$$t_{ij}(s) \sim \frac{c_i c_j}{s_0 - s}$$

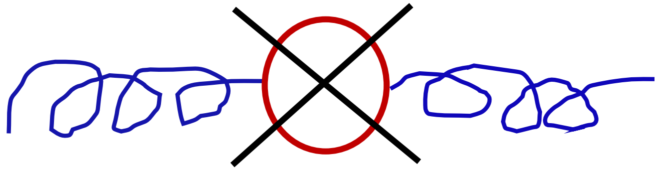
$$\sqrt{s_0} = m_R - i\frac{1}{2}\Gamma_R$$



- With a large coupling a resonance below $h_1^8\eta^8$ threshold

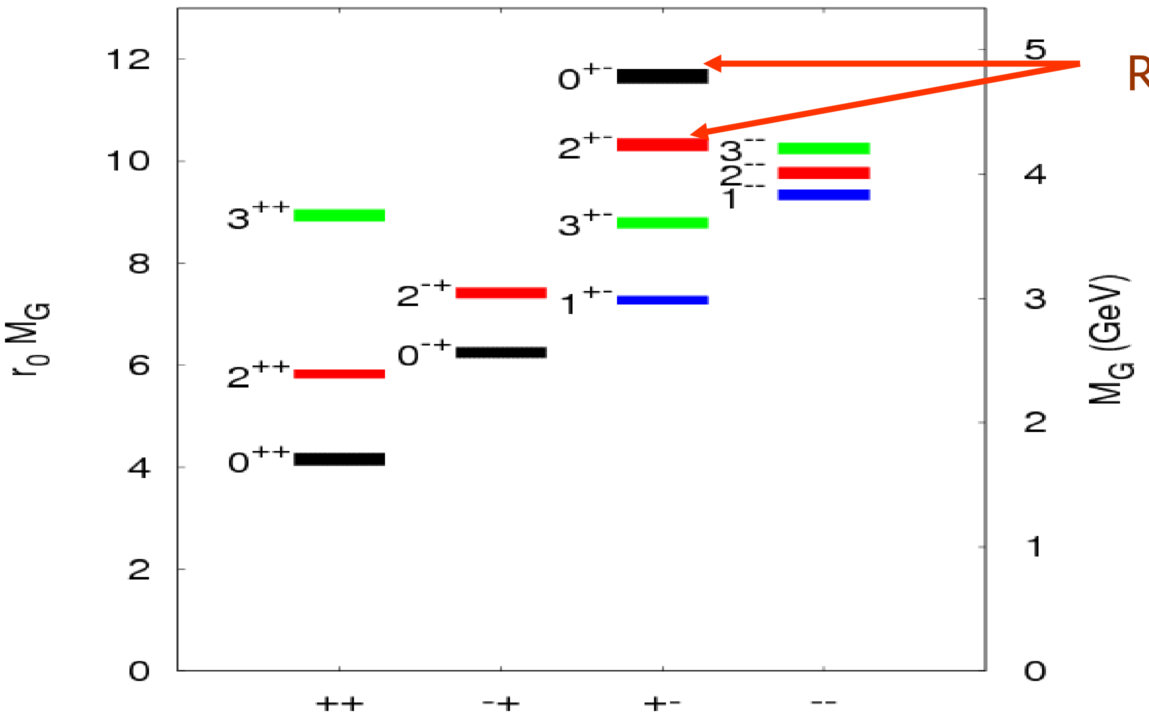
- Flavor octet 1^{-+} state appears as a narrow resonance
- A possible extrapolation to physical kinematics suggests a potentially broad resonance
- Are these channels $\rho\pi, \eta\pi, \eta'\pi$ heavily suppressed?

SU(3) Glueball Spectra

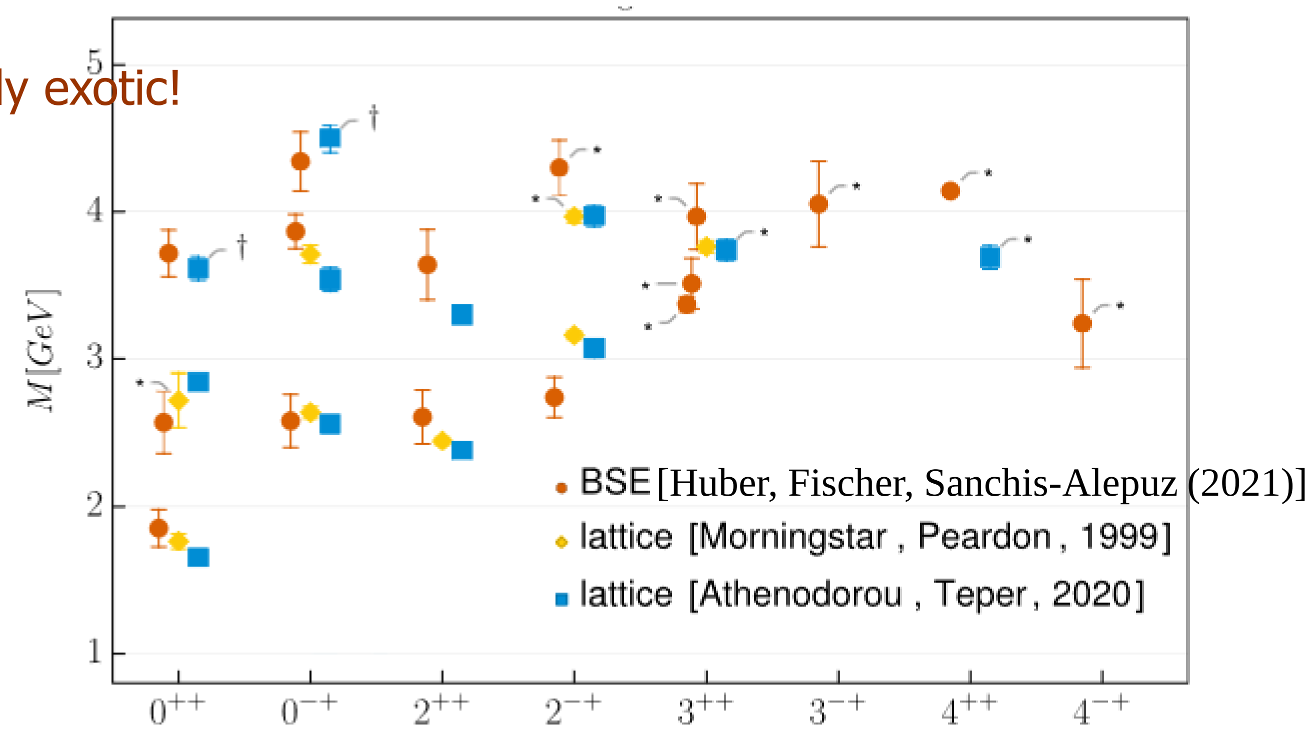


No sea-quarks

Really exotic!



Y. Chen…NM.. et al. Phys. Rev. D73, 014516 (2006)



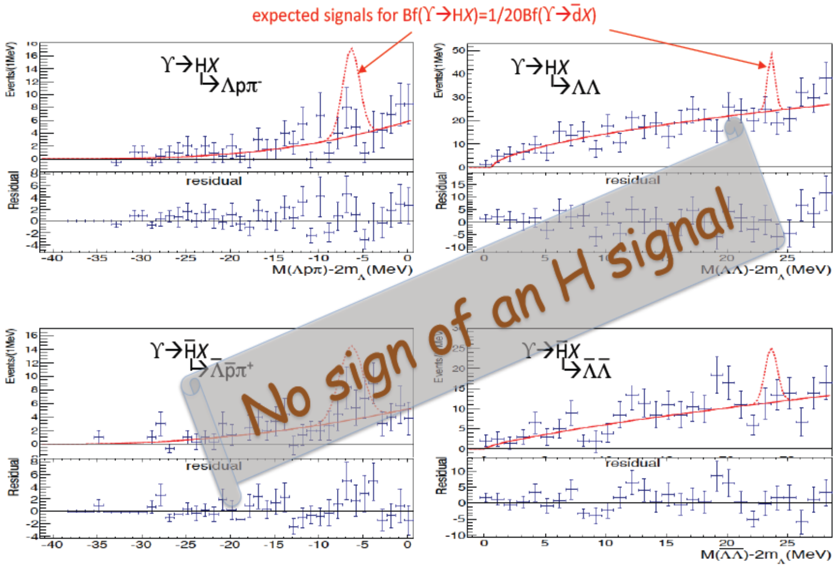
- Signal-to-noise ratios in lattice glueball correlation functions with dynamical quarks are still very poor.
- Multiple channels with glueball, two-quarks and four-quarks with the same quantum numbers need to be addressed together

H Dibaryon

Bound state of two Λ $\Lambda\Lambda\ (udssud)$ Proposed by Jaffe (1976)

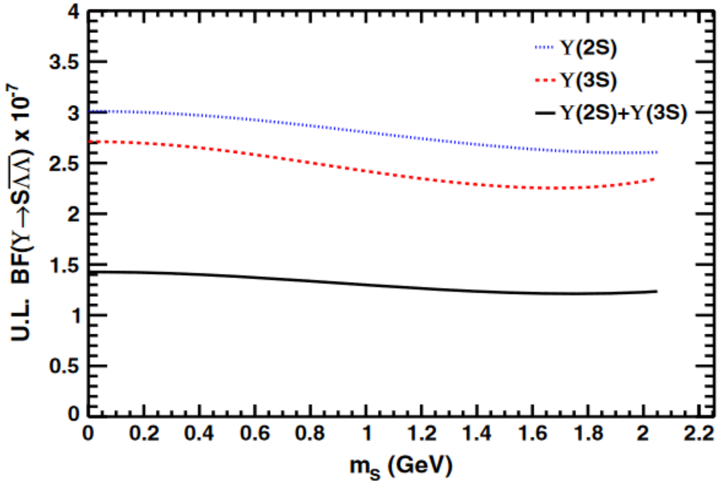
- Has to be below the two proton threshold. Then it will be bound
- If it exists it is extremely stable and could be a candidate for SM dark matter?
  May not be as oxygen may not exist with that!

No H dibaryon

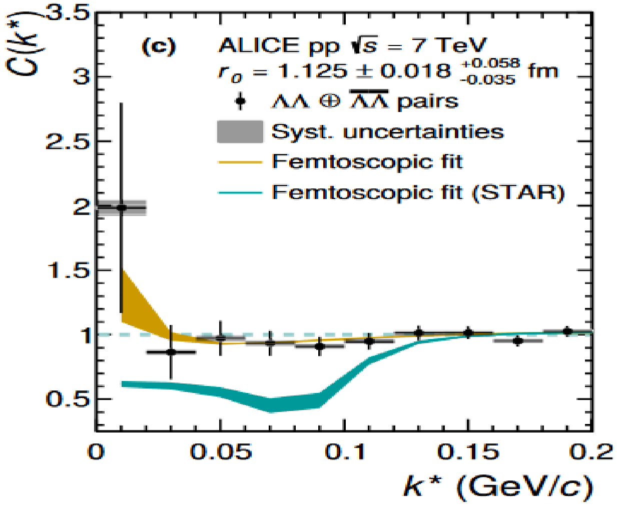


$M(\Lambda p\pi^{-})\text{-}2m_{\Lambda}$ B.H. Kim *et al* (Belle) PRL 110, 222002 (2013) $M(\Lambda\Lambda)\text{-}2m_{\Lambda}$

Belle: PRL 110,222002(2013)

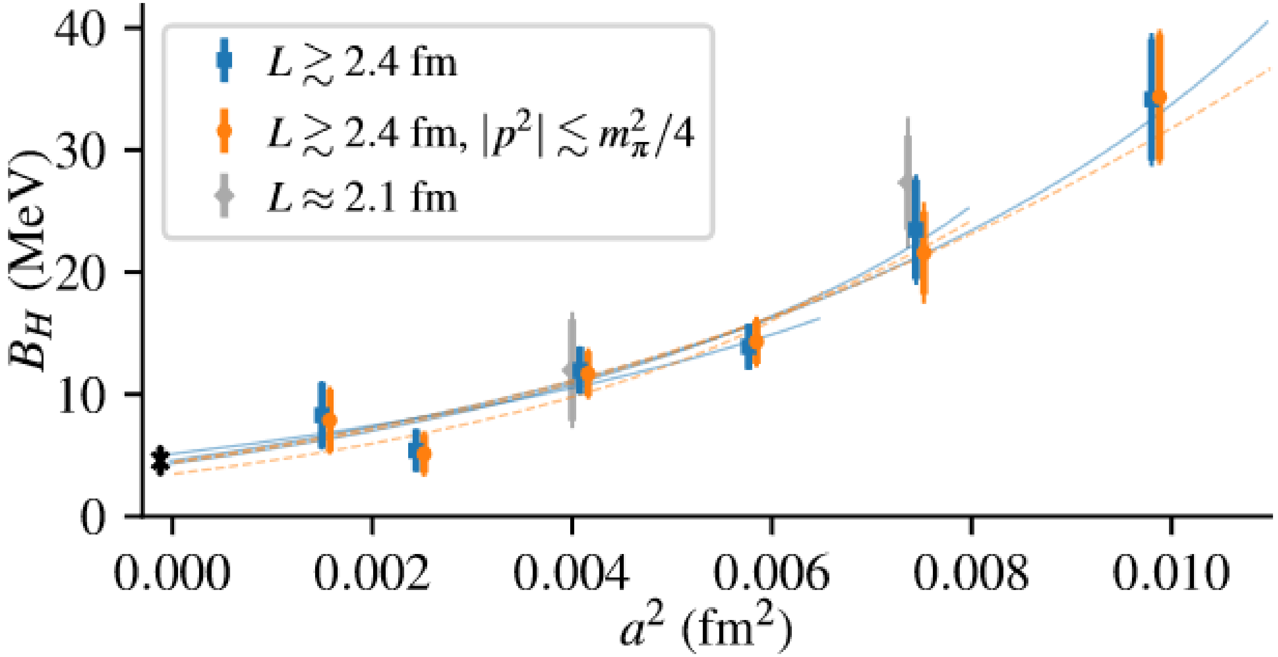


BABAR Collaboration:
Phys. Rev. Lett. 122, 072002 (2019)
No signal is observed
in Υ **decays** (90% confidence limit)



Alice: Phys. Rev. C 99, 024001 (2019)

H-dibaryon at $\mathrm{SU}(3)_{\mathrm{F}}$ symmetric point

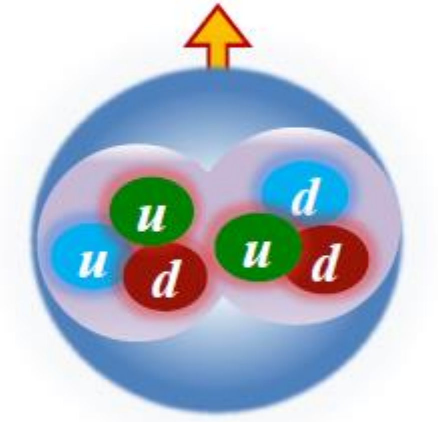


$$\breve{B}_H^{\mathrm{SU(3)_f}} = 4.56 \pm 1.13_{\mathrm{stat}} \pm 0.63_{\mathrm{syst}}\ \mathrm{MeV}.$$

Green et al : *Phys.Rev.Lett.* 127 (2021) 24, 242003

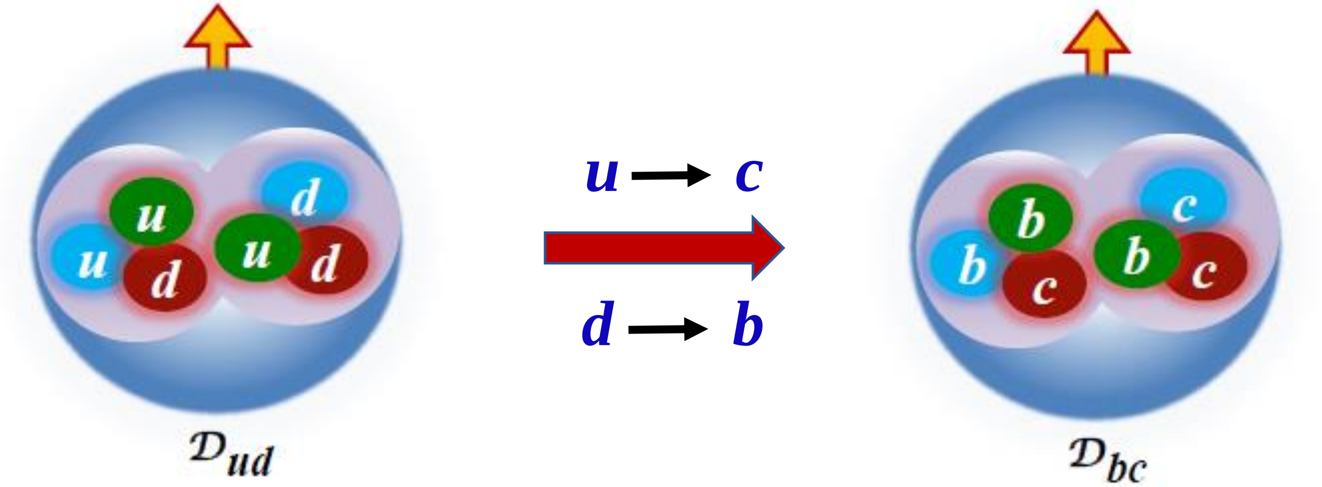
Are there heavy dibaryons?

Deuteron-like heavy dibaryons



$\mathcal{D}_{ud}$

Deuteron-like heavy dibaryons



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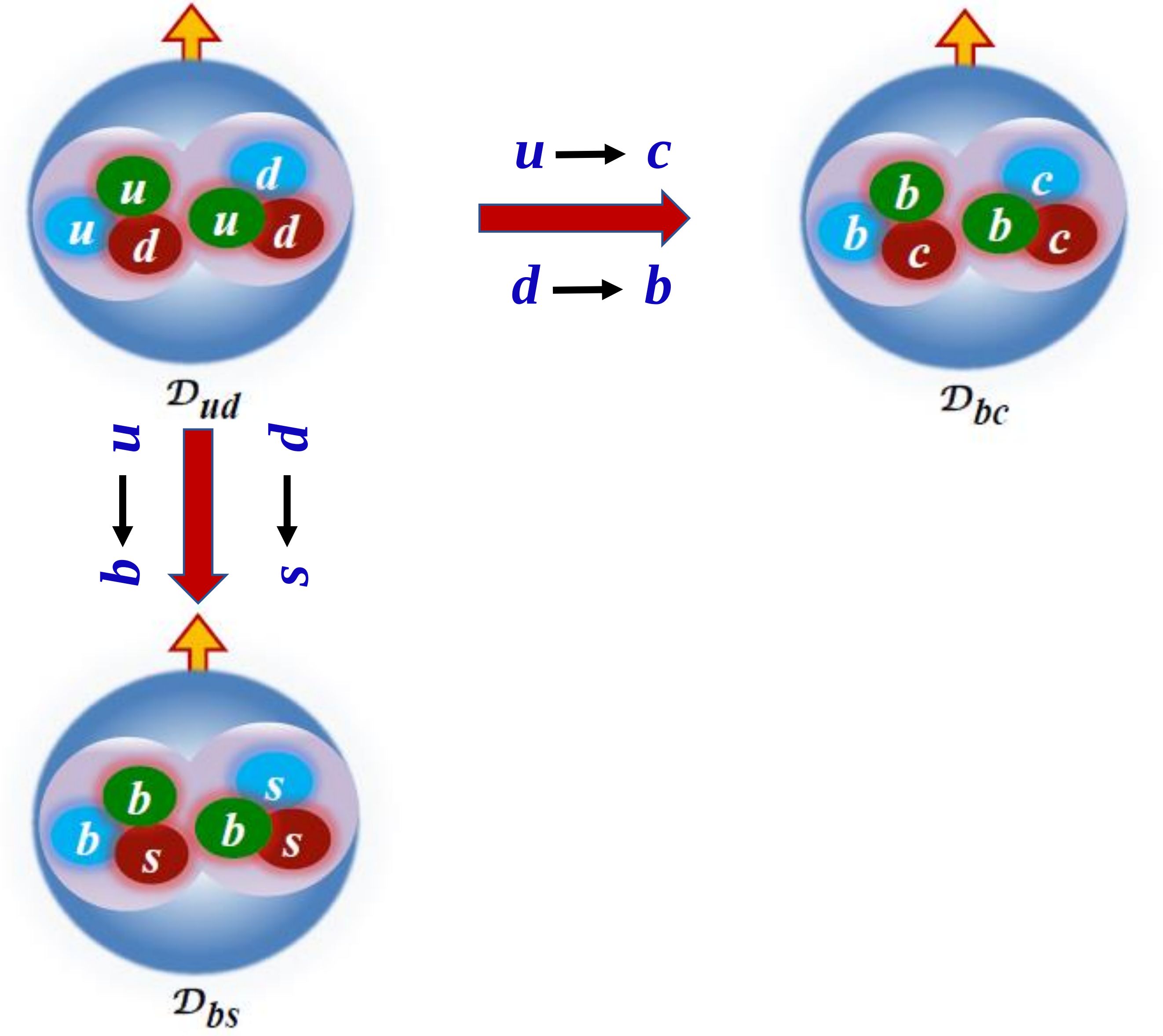
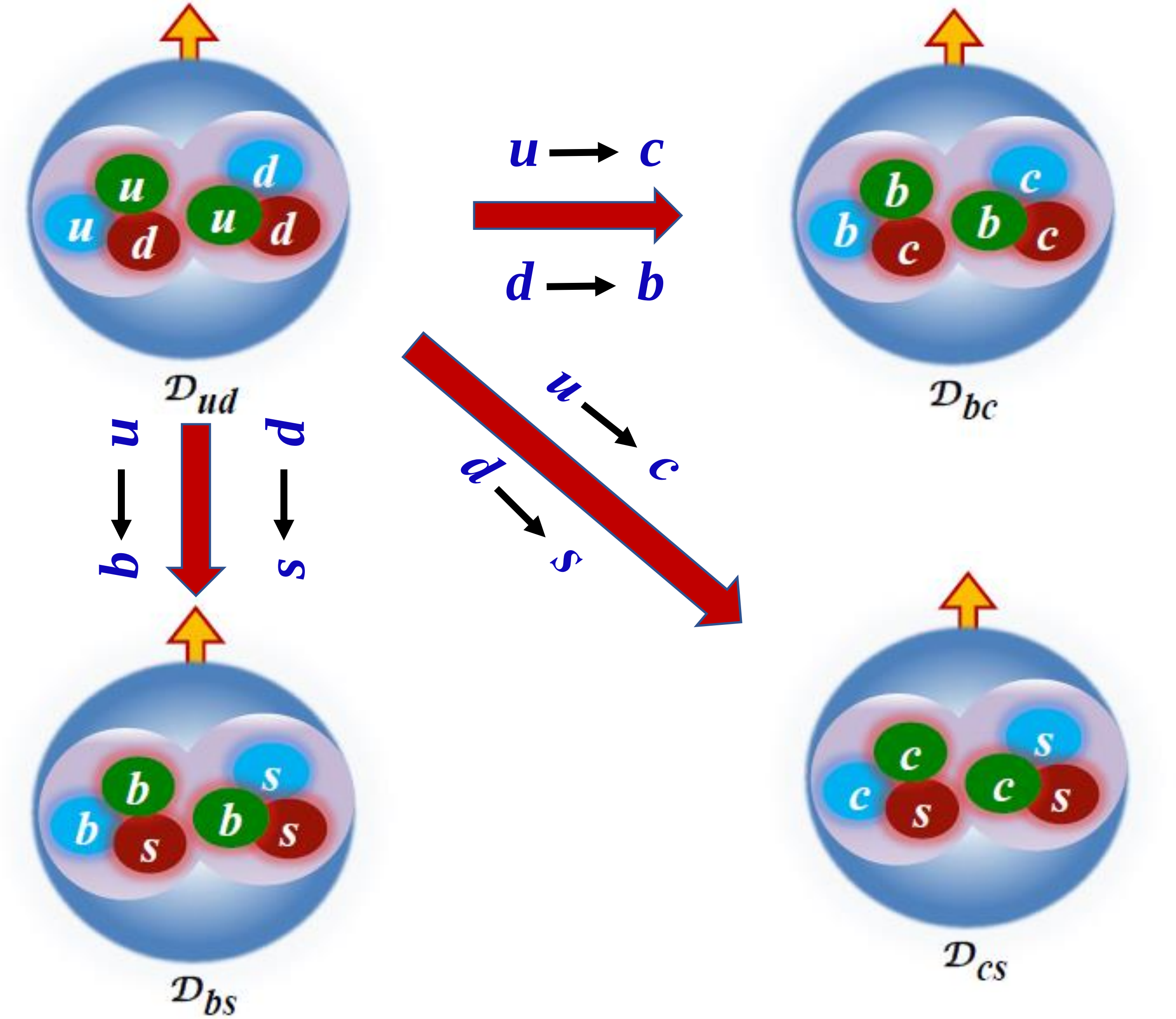
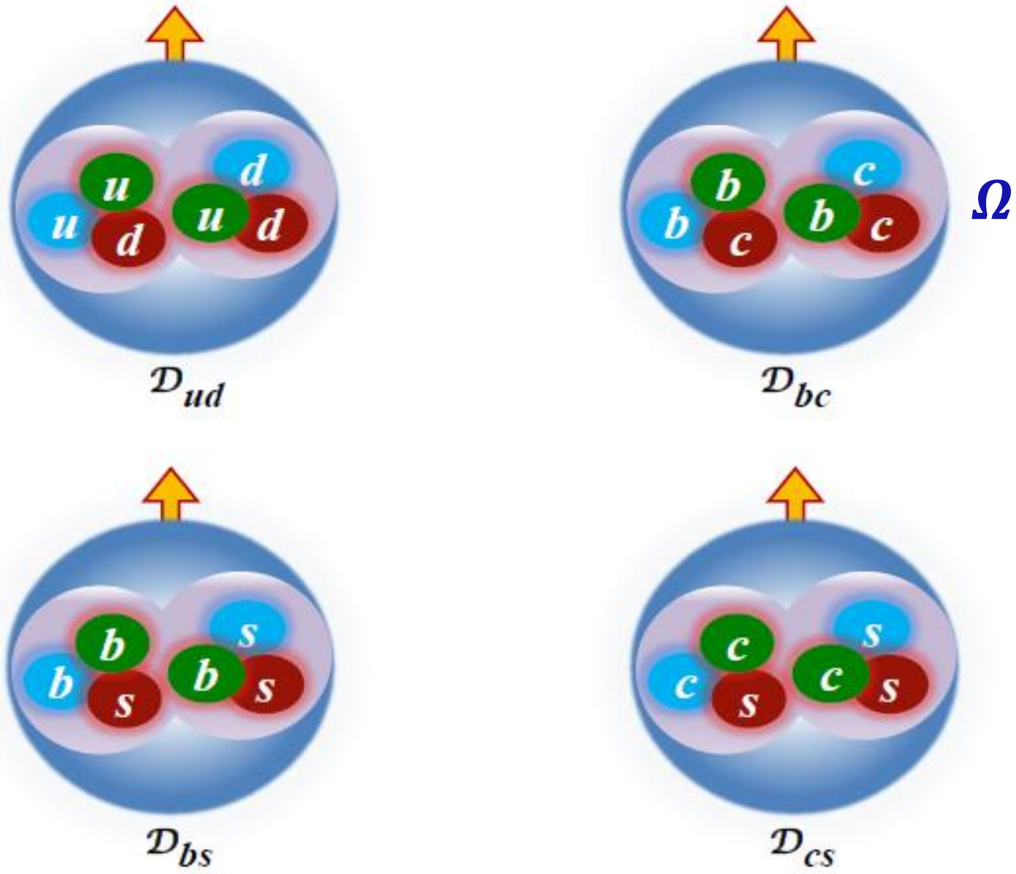


Deuteron-like heavy dibaryons



Deuteron-like heavy dibaryons

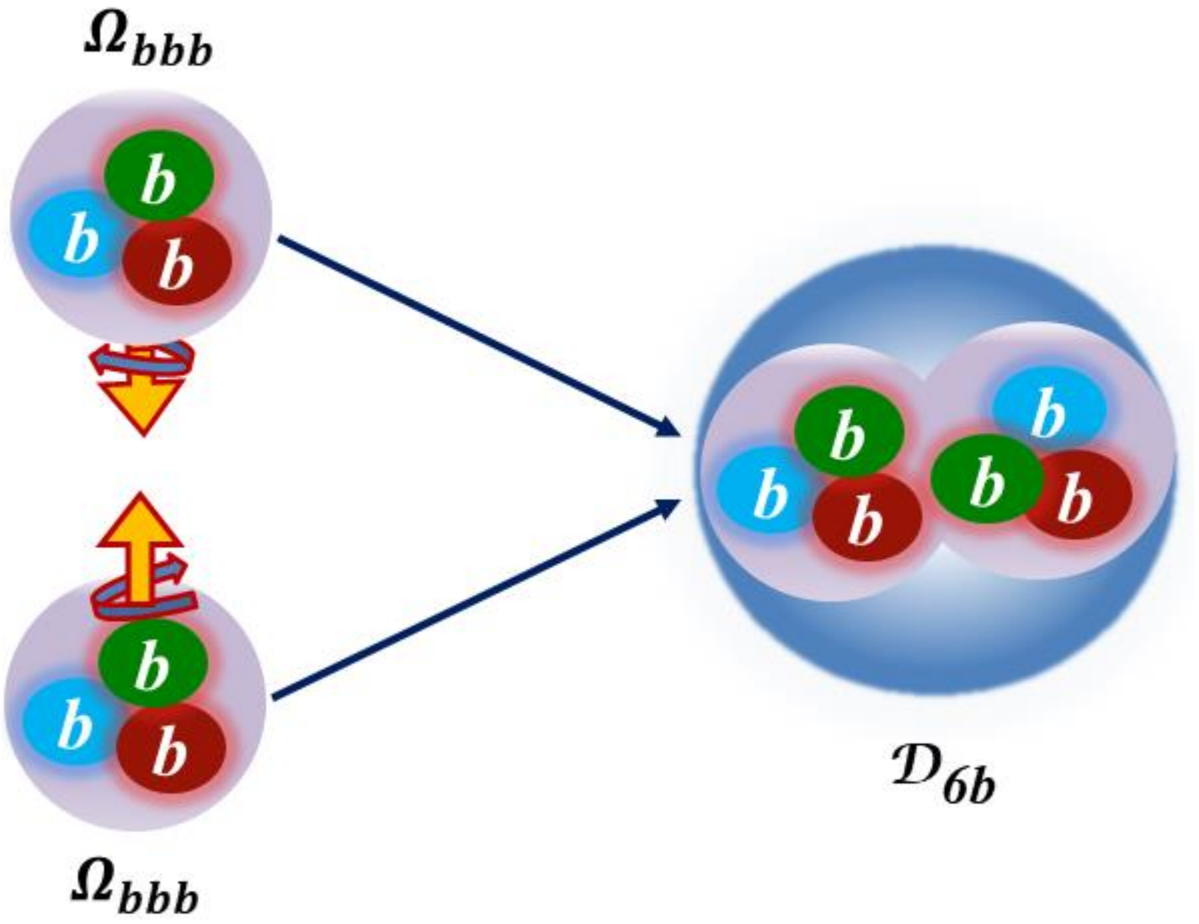
np $\Omega_{ccb}\Omega_{cbb}$



$\Omega_b\Omega_{bb}$ $\Omega_c\Omega_{cc}$

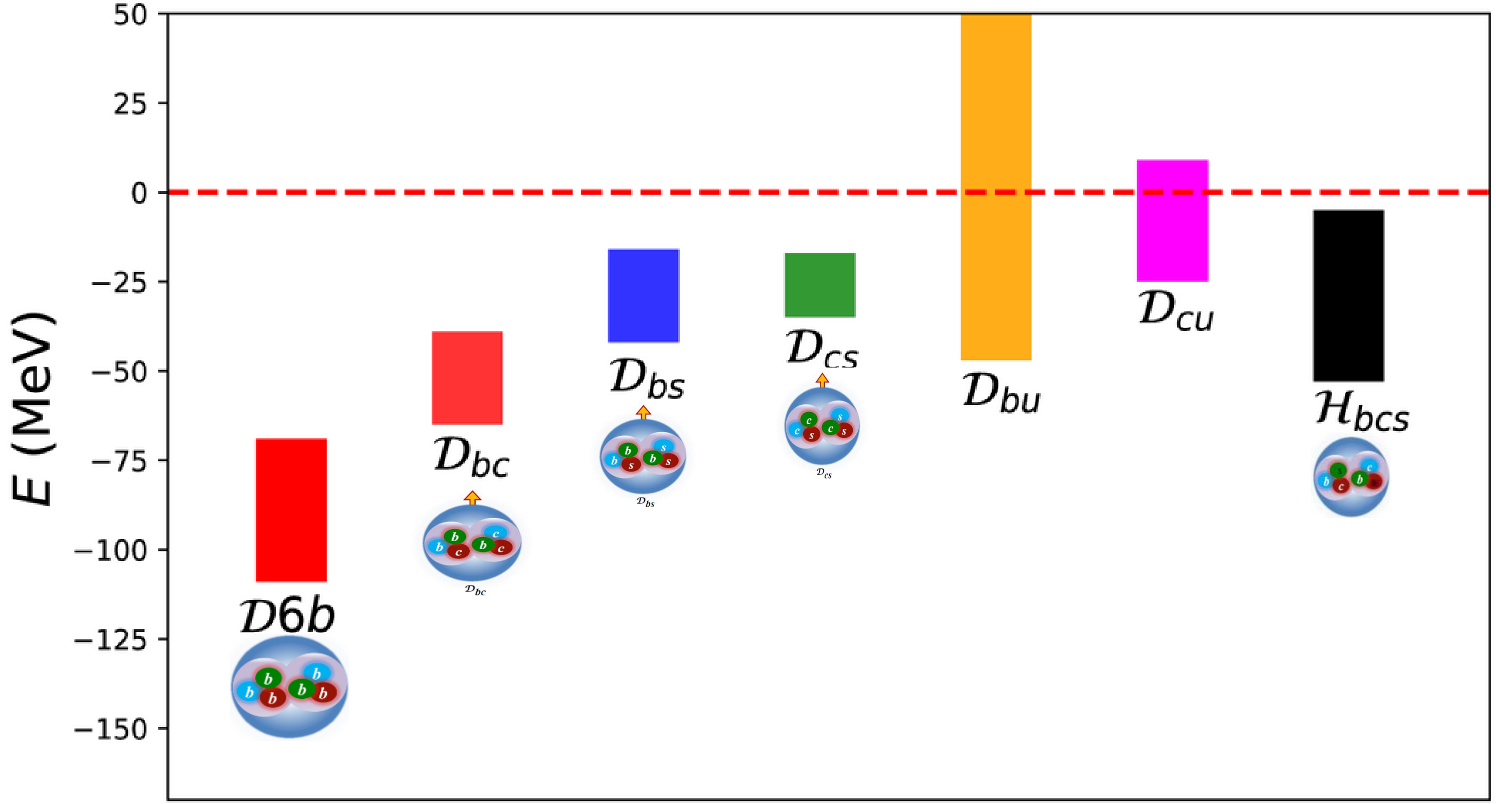
Junnarkar and NM : Phys. Rev. Lett. 123, 162003(2019)

Most beautiful dibaryons!



NM, Padmanath and Chakraborty: arXiv:2205.02862

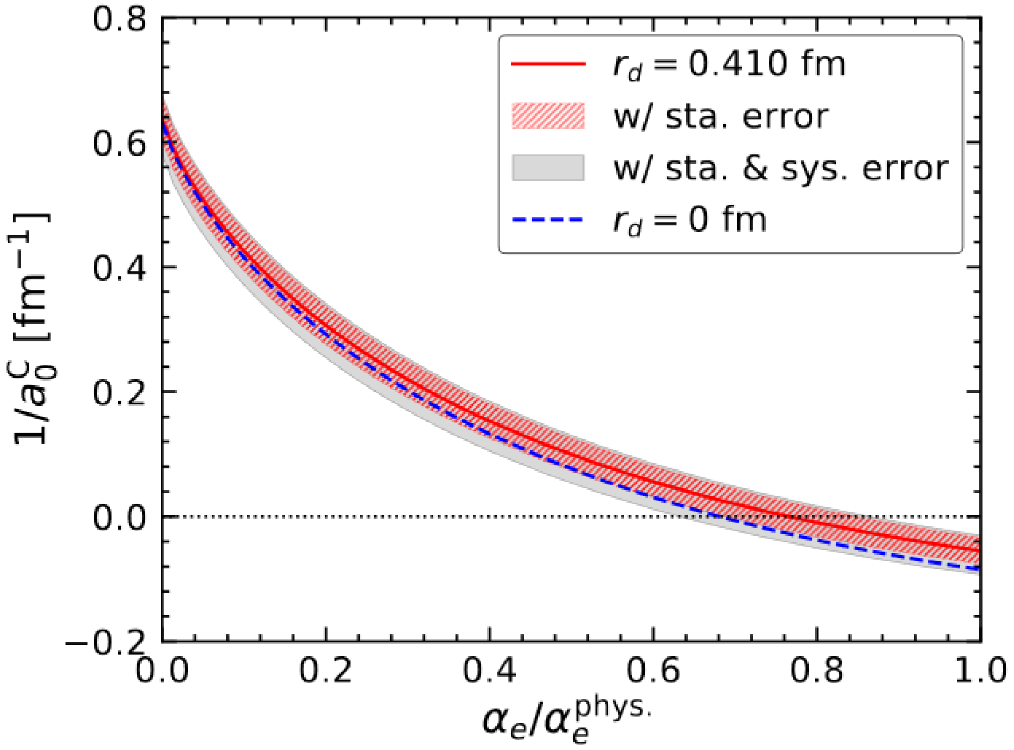
Heavy Dibaryons

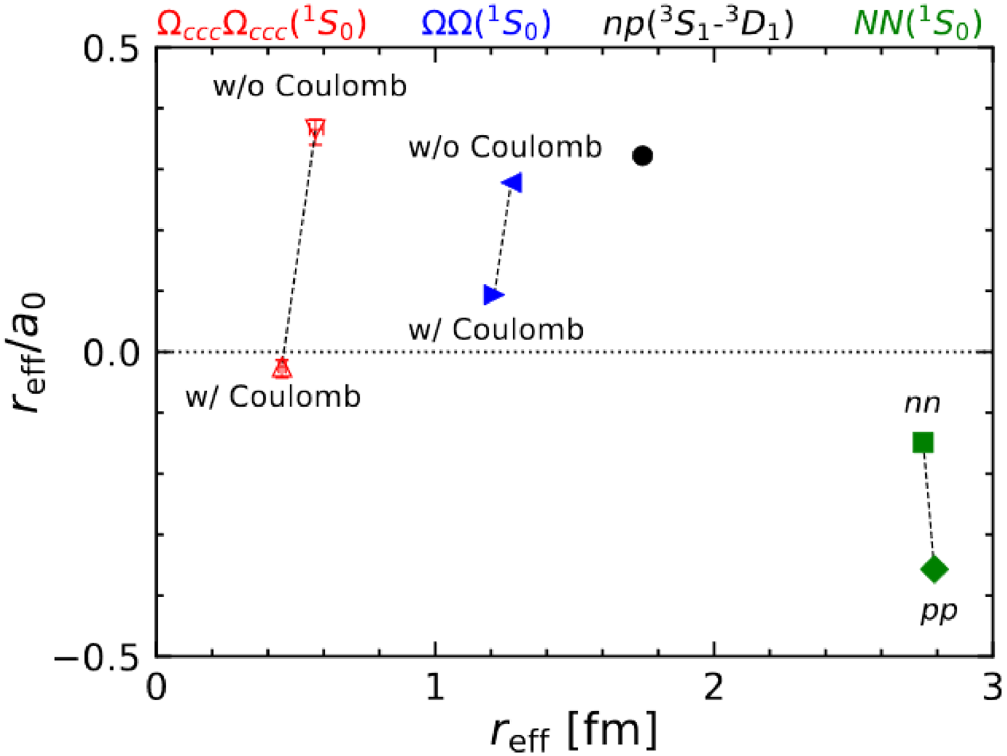


PRL 123,162003 (2019): Junnarkar, NM

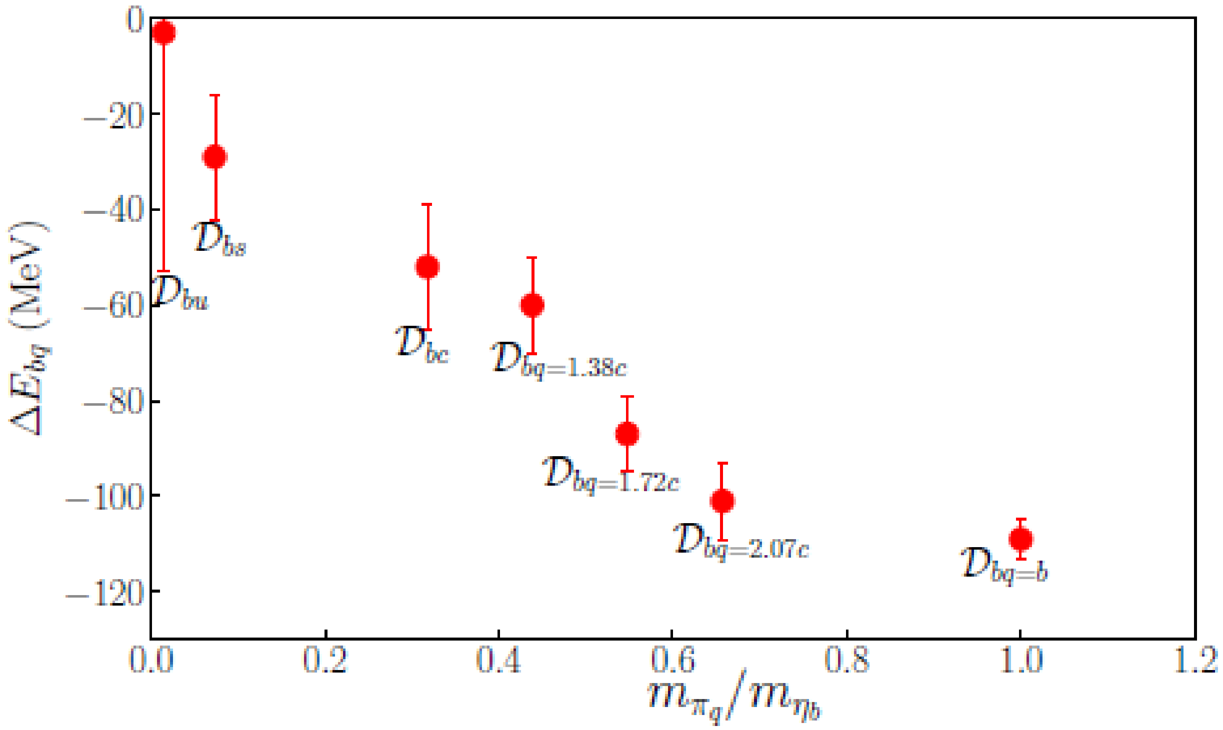
arXiv:2205.02862 NM, Padmanath and Chakraborty

$\Omega_{ccc}\Omega_{ccc}(^1S_0)$

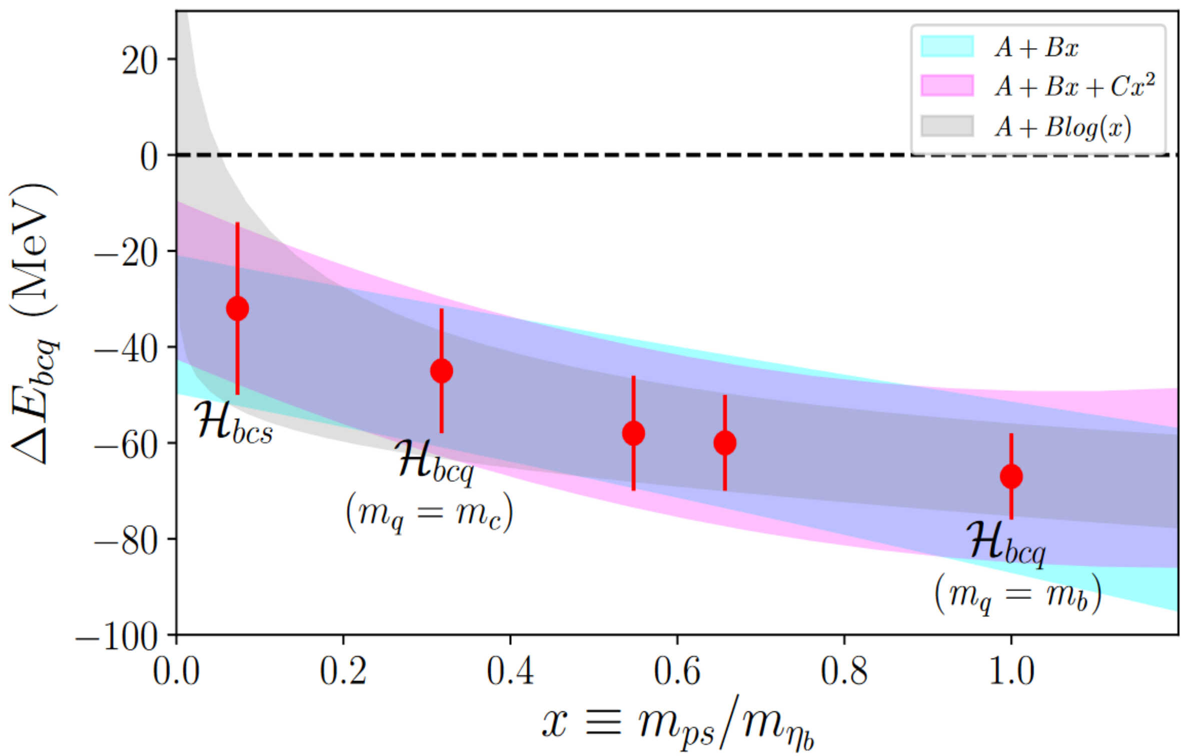




HALQCD:PRL,127, 072003 (2021)



Two-flavored



Three-flavored

- **Heavier the heavy quark mass, stronger the binding**
- **Heavier the light quark mass, stronger the binding**

PRL 123,162003 (2019)

Conclusions and Outlooks

- Lattice QCD provides a rigorous approach to hadron spectroscopy. Tremendous progress has been made in the last decades on the stable and resonance hadron calculations. However, precision comparisons, particularly for unstable hadrons, are not ready yet and that demands much more computational resources.

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- Lattice QCD provides a rigorous approach to hadron spectroscopy. Tremendous progress has been made in the last decades on the stable and resonance hadron calculations. However, precision comparisons, particular for unstable hadrons, are not ready yet and that demands much more computational resources.
- With the available methodologies Lattice QCD is well suited to study exotic multiquark states. However, that needs much more computational resources.
- Multiple lattice QCD studies suggest the existence of a spin-1 deeply bound doubly-bottom four-quark state.
- A recent lattice calculation has found T_{cc} to be a virtual bound state at the higher than physical quark masses.
- Results on T_{bc} are mixed. Because of the closeness to the lowest threshold, a rigorous analysis through finite volume scattering amplitude is essential.
- Results on X,Y,Z-type and pentaquarks, which are around thresholds and have multiple decay modes, need much rigorous lattice study. That requires multiple volumes and lattices with smaller lattice spacings to incorporate both physical heavy and light quarks.

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- Results on X,Y,Z-type four-quarks and pentaquarks, which are around thresholds and have multiple decay modes, need much rigorous lattice study. That requires multiple volumes and lattices with smaller lattice spacings to incorporate both heavy and light quarks at their physical limits.
- Lattice study on the light $\mathbf{1}^{-+}$ spin exotics at $\mathrm{SU(3)_F}$ point has provided impressive results. More precise study demands much more computational resources.
- SU(3) glueball spectra results using lattice QCD is well established. Study of glueballs with sea-quarks requires much more rigorous methods.
- Recent results on H-dibaryon at $\mathrm{SU(3)_F}$ point found to be stable under strong-interactions with a small binding energy.
- One- and two-flavored dibaryons with charm and bottom quarks are found to be stable under strong-interactions and the bottom ones are deeply bound with a pattern that binding energy of a dibaryon increases as the quark masses increase.
- More rigorous lattice QCD results on four and other multiquark configurations are expected to come.